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Simple Extensions with the Minimum Degree Relations of Integral Domains

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Simple Extensions with the Minimum Degree Relations of Integral Domains

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Simple Extensions with the Minimum Degree Relations of Integral Domains

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To the authors' families,
especially to Masako Oda and Sayaka Oda

Preface

This monograph consists of eleven chapters. We study several properties about simple ring-extensions of Noetherian integral domains. The simplest one is a polynomial extension. Many properties of polynomial rings are easily seen. So we may restrict ourselves to algebraic extensions. But arbitrary algebraic simple extensions are not always so simple and are rather complicated. For this reason, we treat simple extensions satisfying a certain condition. Let R be a Noetherian domain and let K be its quotient field. Let α be an element in some algebraic field-extension L of K. The investigation of a simple extension $R[\alpha]$ is equivalent to studying the natural exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0$$

where $\pi : R[X] \to R[\alpha]$ denotes the natural R-algebra homomorphism sending X to α and $R[X]$ is a polynomial ring. Many properties of $R[\alpha]$ are represented by the $\mathrm{Ker}(\pi)$. We say that $R[\alpha]$ is anti-integral over R if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$, where $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ denotes the minimal polynomial of α over K and $I_{[\alpha]}$ denotes the ideal $\bigcap_{i=1}^{d}(R :_R \eta_i)$ in R. Put $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Then $J_{[\alpha]}$ is an ideal of R. If $J_{[\alpha]} \not\subseteq p$ for every prime ideal p of grade one, we say that α is a super-primitive element over R and that $R[\alpha]$ is a super-primitive extension of R. Let $\overline{R}$ denote the integral closure of R in K. If $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$, where $\mathcal{C}(\overline{R}/R) := R :_R \overline{R}$, then we say that α is an ultra-primitive element over R and that $R[\alpha]$ is an ultra-primitive extension of R. We can show the implications: "ultra-primitive $\Rightarrow$ super-primitive $\Rightarrow$ anti-integral." These extensions have the good tool $I_{[\alpha]}$. We also imply that an algebraic element α over a Noetherian Krull domain R is an ultra-primitive element. Thus, ultra-primitiveness (super-primitiveness, anti-integrality) is not so rare and not so special. So we study anti-integral, super-primitive, or ultra-primitive extensions, of R. We divide the inclusion $R \to R[\alpha]$ into $R \to R[\alpha] \cap R[\alpha^{-1}] \to R[\alpha]$. The subring $R[\alpha] \cap R[\alpha^{-1}]$ seems a somewhat interesting subring and it is completely determined if α is anti-integral over R. Note that $R[\alpha] \cap R[\alpha^{-1}]$ and $R[\alpha]$ are birational. We study mainly anti-integral, super-primitive, or ultra-primitive extensions, and we are especially interested in studying the flatness, the integrality, and the unramifiedness of $R[\alpha]$ over R by use of $I_{[\alpha]}$. As will be seen in this monograph,

simple extensions of a Noetherian domain R are not so simple even if they are birationally equal to R.

This monograph owes its existence to the generous efforts of our colleagues and friends, Professors M. Kanemitsu, J. Sato, T. Sugatani, and others, and is based on their joint works with the authors. The authors are deeply grateful to them.

Susumu Oda and Ken-ichi Yoshida

June 30, 2006

Comments

The sources of each section are in the following references:

Chapter 1 §1 [OY1] [K], §2 [OY1] [Ri], §3 [OY1] [Y], §4 [S2] [L] [KY5] §5 [Mc] [S1] [KY3] [OY3], §6 [KY1]

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Chapter 3 §1 [OSuY2] [KY6] [KSaY2] [OY19], §2 [OY6], §3 [TOY], §4 [BOY1] [SaOY3] [OYaY] [OY20]

Chapter 4 §1 [S2], §2 [OSaY3], §3 [KSuY] [OnSuY1] [U], §4 [KY1]

Chapter 5 §1 [OSaY1], §2 [KY1], §3 [OnSuY2]

Chapter 6 §1 [KYJ] [KY2], §2 [KSaY1]

Chapter 7 §1 [OSuY1], §2 [YSaO], §3 [OY4] [OY17] [OY18], §4 [KOY], §5 [OY9]

Chapter 8 §1 [OY14], §2 [OY11] [KSaY3], §3 [SO], §4 [OY13], §5 [OY16]

Chapter 9 §1 [S2], §2 [SaOY1], §3 [OY12], §4 [OY10], §5 [OY15] [OY18]

Chapter 10 §1 [YOSa1], §2 [YOSa1], §3 [YOSa2]

Chapter 11 §1 [OY8], §2 [OKY], §3 [SY1], §4 [OY3]

Convention

Alphabets in brackets refer to the references.

All rings considered in this monograph are assumed to be commutative and have an identity, and our general references for unexplained technical terms are [M1], [M2], and [N].

In particular, throughout this monograph, R denotes an *integral domain*, K denotes *its quotient field*, L denotes an algebraic extension field of K, and $U(R)$ or R^x denotes the set of all units in R unless otherwise specified.

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Chapter 1

Birational Simple Extensions

It is known that if α is a nonzero element in the quotient field K of an integral domain R, then $R\langle\alpha\rangle := R[\alpha] \cap R[\alpha^{-1}]$ is integral over R (*cf.* [K, Ex.4, page 12]). The ring $R\langle\alpha\rangle$ seems rather interesting to us, but we do not recall seeing any other results in the literature about such rings except in a few (e.g., [MR]).

Let R be a Noetherian domain, K be its quotient field, and α be a nonzero element in K, and put $R\langle\alpha\rangle = R[\alpha] \cap R[\alpha^{-1}]$. We call α anti-integral over R if $R\langle\alpha\rangle = R$. In this chapter, we investigate a simple extension $R[\alpha]$ of R with α anti-integral over R, which is called an anti-integral extension of R.

First recall the following fact:

Let A be an over-ring of R (i.e., $R \subseteq A \subseteq K$), which is of finite type over R. Assume that A is quasi-finite over R and R is normal. If the contraction map $\mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$ is surjective, we have $R = A$ (*cf.* [Ra]).

This is known as a special case of the Zariski Main Theorem (ZMT). If R is not integral closed, we can find a counterexample. Even if $\mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$ is surjective, A is not necessarily integral over R (in particular, $R \neq A$). But we can show that if A is $R[\alpha_1, \ldots, \alpha_n]$, each α_i anti-integral over R, then the surjectivity of $\mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$ yields $A = R$. This result is our first motive for writing this chapter (see Section 1.2).

Throughout this chapter, we fix the notation as follows (unless otherwise specified): Let R denote a Noetherian domain, K the quotient field of R, and $\overline{R}$ the integral closure of R in K. For $\alpha \in K$, I_α denotes an ideal $R :_R \alpha := \{\alpha \in R \mid a\alpha \in R\}$ of R, which is called a denominator ideal.

1.1 The Ring $R[\alpha] \cap R[\alpha^{-1}]$

We begin with the following definition.

Definition 1.1.1 For a nonzero element α of K, let

$$R\langle\alpha\rangle := R[\alpha] \cap R[\alpha^{-1}]$$

in K.

The following lemma is seen in [K, Ex.4], but we prove this for the sake of completeness.

Lemma 1.1.2 *Let α is a nonzero element in K. Then $R\langle\alpha\rangle$ is integral over R.*

Proof We know that the integral closure $\overline{R}$ of R can be expressed as $\cap V_\lambda$, where V_λ's are discrete valuation over-rings containing R. Since V_λ is a valuation ring, we have that $\alpha \in V_\lambda$ or $\alpha^{-1} \in V_\lambda$. So $R\langle\alpha\rangle = R[\alpha] \cap R[\alpha^{-1}] \subseteq V_\lambda$ for each V_λ. Hence $R\langle\alpha\rangle \subseteq \overline{R}$. $\square$

The following elementary lemma is useful in the sequel.

Lemma 1.1.3 *Let α be a non-zero element in K. Then the following statements are equivalent:*

(1) *α is integral over R;*

(2) *$R\langle\alpha\rangle = R[\alpha]$;*

(3) *$\alpha \in R\langle\alpha\rangle$.*

Proof (2) $\Leftrightarrow$ (3) is obvious.
(2) $\Leftrightarrow$ (1) follows from Lemma 1.1.2.
(1) $\Rightarrow$ (2): Let α be integral over R. Then

$$\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0 \quad (a_i \in R)$$

So $\alpha = -(a_1 + a_2(\alpha^{-1}) + \cdots + a_n(\alpha^{-1})^{n-1}) \in R[\alpha^{-1}]$, and hence $R[\alpha] \subseteq R[\alpha^{-1}]$. Thus $R[\alpha] \subseteq R[\alpha] \cap R[\alpha^{-1}] = R\langle\alpha\rangle$. $\square$

Definition 1.1.4 Assume that $R = R\langle\alpha\rangle$ for a non-zero $\alpha \in K$. Then we say that α is an *anti-integral element* or that α is *anti-integral* over R. When α is

anti-integral over R, the simple extension $R[\alpha]$ is said to be an *anti-integral extension* of R.

The property of being anti-integral is local-global as is seen in the next lemma.

Lemma 1.1.5 *For a nonzero element α in K, the following statements are equivalent:*

(1) $R\langle\alpha\rangle = R$;

(2) $R_p\langle\alpha\rangle = R_p$ *for each* $p \in \mathrm{Spec}(R)$;

(3) $R_p\langle\alpha\rangle = R_p$ *for each* $p \in \mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) \mid \mathrm{depth}(R_p) = 1\}$.

Proof (1) $\Rightarrow$ (2): Take $\beta \in (R\langle\alpha\rangle)_p$. Then

$$\beta = a_n\alpha^n + \cdots + a_0 = b_m(\alpha^{-1})^m + \cdots + b_0$$

with $a_i, b_j \in R_p$. Choose a common denominator $d \in R$ of a_i's and b_j's such that $d \notin p$. Then $d\beta \in R\langle\alpha\rangle = R$, and hence $\beta \in R_p$.
(2) $\Rightarrow$ (3) is obvious.
(3) $\Rightarrow$ (1): Suppose that $\beta \notin R$ for some $\beta \in R\langle\alpha\rangle$. Then $I_\beta \neq R$. Let p be a prime divisor of I_β. Then $\mathrm{depth}(R_p) = 1$ (*cf.* [Y]). Hence $\beta \in R\langle\alpha\rangle_p = R_p$, so that $I_\beta \not\subseteq p$, which is a contradiction. $\square$

Remark 1.1.6

(1) If $\alpha \in K$ is anti-integral over R, α is also anti-integral over R_p for each $p \in \mathrm{Spec}(R)$.

(2) For a nonzero element $\alpha \in K$, α is anti-integral and integral over R if and only if $\alpha \in R$. $\square$

Theorem 1.1.7 *For a nonzero element $\alpha \in K$, the following statements are equivalent:*

(1) $R[\alpha] \cap \overline{R} = R$ *(i.e., R is integrally closed in $R[\alpha]$);*

(2) *For any nonzero element $\beta \in R[\alpha]$, $R\langle\beta\rangle := R[\beta] \cap R[\beta^{-1}] = R$ (i.e., $\beta \in R[\alpha]$ is anti-integral over R).*

Proof (2) $\Rightarrow$ (1): Suppose that $R[\alpha] \cap \overline{R}$ contains R properly. Then there exists $\beta \in R[\alpha] \cap \overline{R}$ with $\beta \notin R$. Being $\beta \in \overline{R}$ implies that $R\langle\alpha\rangle = R[\beta]$ contains R properly by Lemma 1.1.3. Thus $R\langle\alpha\rangle$ contains R properly.
(1) $\Rightarrow$ (2): Suppose that $R[\alpha] \cap \overline{R} = R$. Since $R \subseteq R\langle\alpha\rangle \subseteq \overline{R} \cap R[\alpha] = R$, we

have $R\langle\alpha\rangle = R$. For any nonzero element $\beta \in R[\alpha]$, we have $R \subseteq R[\beta] \cap \overline{R} \subseteq R[\alpha] \cap \overline{R} = R$. So $R[\beta] \cap \overline{R} = R$ and $R\langle\beta\rangle = R$. $\square$

Remark 1.1.8 For each nonzero element $\alpha \in K$, we have $R\langle\alpha\rangle[\alpha] \cap R\langle\alpha\rangle[\alpha^{-1}] = R\langle\alpha\rangle$, i.e., α is anti-integral over $R\langle\alpha\rangle$. In other words, $R[\alpha]$ is an anti-integral extension of $R\langle\alpha\rangle$. In fact, this is implied as follows: $R\langle\alpha\rangle \subseteq R\langle\alpha\rangle[\alpha] \cap R\langle\alpha\rangle[\alpha^{-1}] = (R[\alpha] \cap R[\alpha^{-1}])[\alpha] \cap (R[\alpha] \cap R[\alpha^{-1}])[\alpha^{-1}] \subseteq R[\alpha] \cap R[\alpha^{-1}] = R\langle\alpha\rangle$. $\square$

Lemma 1.1.9 *Let α be an anti-integral element in K over R. Let $\beta = u\alpha + a$ $(u, a \in R)$ with u a unit in R. Then β is an anti-integral element over R.*

Proof It is easy to see that $u\alpha$ is an anti-integral element over R. So we may assume that $u = 1$ and $\beta = \alpha + a$ $(a \in R)$. Let

$$0 \longrightarrow I \longrightarrow R[X] \xrightarrow{\varphi} R[\beta] \longrightarrow 0$$

be an exact sequence, where $R[X]$ is a polynomial ring and $\varphi(X) = \beta$. By [M1], it suffices to show that I has a linear basis, that is, $I = \sum(c_i X - d_i)R[X]$ with $\beta = d_i/c_i$ $(c_i, d_i \in R)$. Take $f(X) \in I \subseteq R[X]$. Putting $Y = X - a$, let $g(Y) = f(X)$. Since $f(\beta) = g(\alpha) = 0$ as α is anti-integral over R, we have $g(Y) = \sum(a_i Y - b_i)h_i(Y) = \sum(a_i(X - a) - b_i)h_i(X - a) = \sum(a_i - X - (b_i + a_i a))h_i(X - a)$, and $(b_i + a_i a)/a_i = b_i/a_i + a = \alpha + a = \beta$. Thus I has a linear basis, which means that β is anti-integral over R. $\square$

Proposition 1.1.10 *For a nonzero element $\alpha \in K$, if $R\langle\alpha\rangle$ contains R properly then there exists $r \in R$ such that $r\alpha \in R\langle\alpha\rangle \setminus R$.*

Proof Let β be an element in $R\langle\alpha\rangle \setminus R$. Then

$$(*) \qquad \beta = a_n\alpha - n + \cdots + a_0 = b_m(\alpha^{-1})^m + \cdots + b_0$$

with $a_i, b_j \in R$. Changing β and the expression $(*)$, we may assume that n is the smallest among such n's. If $n = 1$, then there is nothing to prove. Suppose $n > 1$. We can assume that $a_0 = 0$. Thus

$$\beta/\alpha = a_n\alpha^{n-1} + \cdots + a_1 = b_m(\alpha^{-1})^{m+1} + \cdots + b_0(\alpha^{-1}) \in R\langle\alpha\rangle$$

By the choice of n, we have $\beta/\alpha \in R$, and hence we conclude that $\beta = r\alpha$ with $r \in R$. $\square$

Corollary 1.1.11 *For a nonzero element $\alpha \in K$, let $I_\alpha = q_1 \cap \ldots \cap q_t$ be a primary decomposition. If $\sqrt{q_i} \notin \mathrm{Ass}_R(\overline{R}/R)$ for each i, then α is anti-integral over R.*

Proof Suppose that $R\langle\alpha\rangle$ contains R properly. Then there exists $r \in R$ such that $r\alpha \in R\langle\alpha\rangle \setminus R$ by Proposition 1.1.10, since $I_{r\alpha} = I_\alpha :_R r = (q_1 : r) \cap \ldots \cap (q_t : r)$. Since $r\alpha$ is integral over R (Lemma 1.1.2), there exists i such that $\sqrt{q_i} \in \mathrm{Ass}_R(\overline{R}/R)$. $\square$

Remark 1.1.12 Let $\alpha \in K$ and let S be a multiplicatively closed subset of R. Then $S^{-1}(R\langle\alpha\rangle) = S^{-1}R[\alpha] \cap S^{-1}R[\alpha^{-1}]$. If an over-ring A is an anti-integral extension of R, $S^{-1}A$ is an anti-integral extension of $S^{-1}R$, and a polynomial extension $A[X]$ is an anti-integral extension of $R[X]$. $\square$

1.2 Anti-Integral Extension and Flat Simple Extensions

To begin with, we give the following definition.

Definition 1.2.1 Let $A := R[\alpha]$, α a nonzero element in K. For $p \in \mathrm{Spec}(R)$, we denote $A_p = R_p[\alpha]$. We say that an extension A/R is a *blowing-up* at p if the following two conditions are satisfied:

(i) $pA_p \cap R_p = pR_p$,

(ii) A_p/pA_p is isomorphic to a polynomial ring $(R_p/pR_p)[T]$.

Lemma 1.2.2 *Let (R, m) be a local domain and put $A := R[\alpha]$, $\alpha \in K$ ($\alpha \neq 0$). Assume that $mA \cap R = m$. In this case we have a canonical injection $R/m \longrightarrow A/mA = (R/m)[\overline{\alpha}]$. Then*

(1) *A/R is a blowing-up at m if and only if $\overline{\alpha}$ is transcendental over R/m (where $\overline{\alpha}$ denotes the residue class of α in A/mA).*

(2) *Assume that R contains an infinite field k. If A/R is not a blowing-up at m, then for some $\mu \in k$, $1/(\alpha - \mu)$ is integral over R and i is integral and j is a localization ($flat$) in the following sequence:*

$$R \xrightarrow{i} R\langle\alpha - \mu\rangle = R[1/(\alpha - \mu)] \xrightarrow{j} R[\alpha] = R[\alpha - \mu, 1/(\alpha - \mu)]$$

Proof (1) is obvious.
(2) By the preceding (1), $\overline{\alpha}$ is algebraic over R/m. So we have

$$(**) \quad \overline{\alpha}^k(\overline{\alpha}^\ell + \lambda_1'\overline{\alpha}^{\ell-1} + \cdots + \lambda_\ell') = 0 \quad (\lambda_i' \in R/m,\ \lambda_\ell' \neq 0)$$

If $k > 0$, we can find $\mu \neq 0$ in the infinite field k such that $\mu^\ell + \lambda_1'\mu^{\ell-1} + \cdots + \lambda_\ell' \neq 0$. So putting $\beta = \alpha - \mu$, from $(**)$ we have an algebraic relation:

$$\overline{\beta}^n + \lambda_1\overline{\beta}^{n-1} + \cdots + \lambda_n = 0 \quad (\lambda_n \neq 0,\ \lambda_i \in R/m)$$

Let a_i denote the representative of λ_i in R. Then we have

$$\beta^n + a_1\beta^{n-1} + \cdots + a_n \in mR[\beta]$$

Since $\overline{a}_n = \lambda_n \neq 0$, we have $a_n \notin m$. By the expression of an element in $mR[\beta]$, we get

$$b_\ell\beta^\ell + \cdots + b_0 = 0$$

with $b_i \in R$ and $\overline{b}_0 = \overline{a}_n = \lambda_n \neq 0$. Hence b_0 is a unit in R. Multiplying $1/b_0$,

$$c_\ell\beta^\ell + \cdots + c_1\beta + 1 = 0$$

which means that β^{-1} is integral over R. By Lemma 1.1.3, $R\langle\beta\rangle = R\langle\beta^{-1}\rangle = R[\beta^{-1}]$ and $R \xrightarrow{i} R\langle\beta^{-1}\rangle = R[\beta^{-1}] \xrightarrow{j} R[\beta, \beta^{-1}]$, where i is integral and j is a localization (hence flat). $\square$

Definition 1.2.3 Let $R \longrightarrow A$ be a ring extension and $p \in \mathrm{Spec}(R)$. We say that p is *contracted* from A if $pA \cap R = p$.

It is easy to see that $p \in \mathrm{Spec}(R)$ is contracted from A if and only if $P \cap R = p$ for some $P \in \mathrm{Spec}(A)$ if and only if $pA_p \cap R_p = PR_p$. If $A = R[\alpha]$ is a blowing-up at $p \in \mathrm{Spec}(R)$, then p is contracted from A.

Lemma 1.2.4 *Let (R, m) be a local domain and put $A = R[\alpha]$ with a nonzero $\alpha \in K$. Then*

(1) *m is not contracted from A if and only if $mA = A$;*

(2) *if m is not contracted from A, then α^{-1} is integral over R and $R \xrightarrow{i} R\langle\alpha\rangle = R[\alpha^{-1}] \xrightarrow{j} R[\alpha] = R[\alpha, \alpha^{-1}]$, where i is integral and j is flat.*

Proof (1) is obvious.
(2) Suppose that $mA = A$. Then

$$1 = a_0 + a_1\alpha + \cdots + a_n\alpha^n \qquad (a_i \in m)$$

Since $1 - a_0$ is a unit in R, α^{-1} is integral over R. Hence noting that $\alpha^{-1} \in R\langle\alpha\rangle \subseteq R[\alpha]$, we have a sequence

$$R \xrightarrow{i} R\langle\alpha\rangle = R[\alpha^{-1}] \xrightarrow{j} R[\alpha] = R[\alpha, \alpha^{-1}]$$

where i is integral and j is a localization. $\square$

Proposition 1.2.5 *Let (R, m) be a local domain and let α be a nonzero element in K which is anti-integral over R. Assume that $R[\alpha]$ contains R properly. Then $mR[\alpha] = R[\alpha]$ if and only if $R = R\langle\alpha\rangle = R[\alpha^{-1}] \xrightarrow{j} R[\alpha]$, where j is flat.*

Proof ($\Rightarrow$): By Lemma 1.2.4, α^{-1} is integral over R and $R\langle\alpha\rangle = R\langle\alpha^{-1}\rangle = R[\alpha^{-1}] \xrightarrow{j} R[\alpha]$, where j is flat. Since α is anti-integral, we have $R\langle\alpha\rangle = R$.
($\Leftarrow$): By assumption, $\alpha \notin R$. Since $\alpha^{-1} \in R$, we have $\alpha^{-1} \in m$, which implies that $mR[\alpha] = R[\alpha]$. $\square$

The ring-extension $R[\alpha]$ of R is characterized by an ideal I_α as is seen in the following proposition.

Theorem 1.2.6 *Let $\alpha \in K$ be anti-integral over R and let p be a prime ideal of R. Assume that R contains an infinite field k. Then*

(1) *$I_\alpha \not\subseteq p$ if and only if $R_p = R_p[\alpha]$;*
(2) *$I_\alpha \subseteq p$ but $\alpha I_\alpha \not\subseteq p$ if and only if $R_p \neq R_p[\alpha]$, and p is not contracted from $R[\alpha]$;*
(3) *$I_\alpha + \alpha I_\alpha \subseteq p$ if and only if $R[\alpha]$ is blowing-up at p over R.*

Proof (1): $I_\alpha \not\subseteq p \Leftrightarrow \alpha \in R_p \Leftrightarrow R_p[\alpha] = R_p$.
(2) ($\Rightarrow$): Since $\alpha I_\alpha \not\subseteq p$, there exists $b \in \alpha I_\alpha$ but $b \notin p$. Hence $\alpha = b/a$ for some $a \in I_\alpha \subseteq p$. So $a\alpha = b \in pR_p[\alpha]$ and hence $pR[\alpha] \cap R$ contains p properly.
($\Leftarrow$): By Lemma 1.2.4, $R_p \xrightarrow{i} R_p[\alpha^{-1}] \xrightarrow{j} R_p[\alpha^{-1}]$, where i is integral and j is flat, and α^{-1} is integral over R_p. Since α is anti-integral over R_p, we have $R_p = (R\langle\alpha\rangle)_p = (R\langle\alpha^{-1}\rangle)_p = R_p[\alpha^{-1}]$. Hence $\alpha^{-1} \in R_p$, which implies that $\alpha I_\alpha = I_{\alpha^{-1}} \not\subseteq p$. Since $pR_p[\alpha] = R_p[\alpha]$, we have $R_p[\alpha]$ contains R_p properly, and hence $I_\alpha \subseteq p$.
(3) ($\Rightarrow$): Since $R_p[\alpha]/pR_p[\alpha] = (R_p/pR_p)[\overline{\alpha}]$, we must show that $\overline{\alpha}$ is transcendental over R_p/pR_p. Suppose the contrary. We have by Lemma 1.2.2 that β^{-1} is integral over R_p with $\beta = \alpha - \mu$ for some $\mu \in k$, and β is anti-integral over R by Lemma 1.1.9. So $\beta^{-1} \in R_P$. Hence $\beta I_\beta = I_{\beta^{-1}} \not\subseteq p$. But it is easy to

see that $I_\alpha + \alpha I_\alpha = I_\alpha(R + \alpha R) = I_\beta(R + \beta R) = I_\beta + \beta I_\beta$, which yields that $I_\beta + \beta I_\beta \subseteq p$, a contradiction.
($\Leftarrow$): Suppose that $I_\alpha + \alpha I_\alpha \not\subseteq p$. If $I_\alpha \not\subseteq p$, then $R_p = R_p[\alpha]$ and $R[\alpha]$ is not a blowing-up at p by (1). If $I_\alpha \subseteq p$ and $\alpha I_\alpha \not\subseteq p$, then $pR_p[\alpha] \cap R_p \neq pR_p$ by (2). Thus $R[\alpha]$ is not a blowing-up at p by Definition 1.2.1. Therefore in any case $R[\alpha]$ is not a blowing-up at p. ∎

The following result is shown in [Ri]:

Lemma 1.2.7 *Let A be an integral domain (not necessarily Noetherian), L its quotient field, and $\alpha = b/a \in L$ ($a,b \in A, ab \neq 0$). If $A[\alpha]$ is flat over A and A is integrally closed in L, the ideal $a(1, \alpha)A = (a,b)A$ is invertible.*

Proof Since $A \subseteq A[\alpha]$ is flat, we have that $(b : Aa)A[\alpha] = A[\alpha]$, i.e., there exist $h_j \in (b :_A a)$ such that $h_n a^n/y^n + h_{n-1}a^{n-1}/y^{n-1} + \cdots + h_1 a/b + h_0 = 1$, for some nonnegative integer n. Multiplying by b^n/a^n, we get that b/a is integral over $A[1/s]$, where $s = h_0 - 1$, and so $b/a \in A[1/s]$ since $A[1/s]$ is integrally closed. Therefore $b/a = h/(h_0 - 1)^t$ for some $h \in A$ and some nonnegative integer t, and so $ha/b + ch_0 = 1$ for some $c \in A$, i.e., $(h/b)a + (ch_0/b)b = 1$, where h/b and $ch_0/b \in (a, b)^{-1}$. ∎

Theorem 1.2.8 *Let $\alpha \in K$ be a nonzero element. Then*

(1) *If $I_\alpha + \alpha I_\alpha = R$ then $R[\alpha]$ is flat over R.*
(2) *Assume that $\alpha = b/a$ ($a, b \in R$) is anti-integral over R. Then the following statements are equivalent:*
   (2.i) *$R[\alpha]$ is flat over R;*
   (2.ii) *$I_\alpha + \alpha I_\alpha = R$;*
   (2.iii) *$(a,b)R$ is an invertible ideal of R.*

Proof (1) Since $I_\alpha(R + \alpha R) = R$, a fractional ideal $R + \alpha R$ is invertible. Since $R + \alpha R \simeq a(R + \alpha R) = (a, b)R$ is invertible, so $R[\alpha]$ is flat (*cf.* [Ri]).
(2) (2.iii) $\Rightarrow$ (2.i) follows from [Ri]. (2.ii) $\Rightarrow$ (2.iii) is shown above. (2.i) $\Rightarrow$ (2.ii): Note first that $p \supseteq I_\alpha + \alpha I_\alpha$ if and only if $R[\alpha]$ is a blowing-up at p over R by Theorem 1.2.6. Since $R[\alpha]$ is flat over R, $R[\alpha]$ is not a blowing-up at any $p \in \mathrm{Spec}(R)$ over R. Thus $I_\alpha + \alpha I_\alpha = R$. ∎

Proposition 1.2.9 *Assume that $\alpha \in K$ is an anti-integral element over R. Then $R[\alpha]$ is flat over R if and only if $R[\alpha^{-1}]$ is flat over R.*

Proof By Theorem 1.2.8, $R[\alpha]$ is flat over $R \Leftrightarrow I_\alpha + I_{\alpha^{-1}} = I_\alpha + \alpha I_\alpha = R \Leftrightarrow I_\alpha + I_{\alpha^{-1}} = (\alpha^{-1})I_{\alpha^{-1}} + I_{\alpha^{-1}} = R$ (Note that $\alpha I_\alpha = I_{\alpha^{-1}}$) $\Leftrightarrow R[\alpha^{-1}]$ is flat over R. ∎

Lemma 1.2.10 *Let $\alpha \in K$ ($\alpha \neq 0$) and $P \in \mathrm{Spec}(R\langle\alpha\rangle)$. Put $p = P \cap R$. Then either $pR[\alpha] \neq R[\alpha]$ or $pR[\alpha^{-1}] \neq R[\alpha^{-1}]$ holds.*

Proof Suppose that $pR[\alpha] = R[\alpha]$. Then we have $(R\langle\alpha\rangle)_p = R_p[\alpha^{-1}]$ by Lemma 1.2.4 since $pR_p[\alpha] = R_p[\alpha]$. Hence $(R\langle\alpha\rangle)_p \underset{\neq}{\supset} P(R\langle\alpha\rangle)_p \supseteq p(R\langle\alpha\rangle)_p = pR_p[\alpha^{-1}]$ and consequently we have $pR_p[\alpha^{-1}] \neq R_p[\alpha^{-1}]$. Thus $pR[\alpha^{-1}] \neq R[\alpha^{-1}]$. ∎

The next result is well known (*cf.* [Ri]).

Lemma 1.2.11 *Let A be an over-ring of R. Then A is flat over R if and only if $A_P = R_{P\cap R}$ for each $P \in \mathrm{Spec}(A)$.*

Proposition 1.2.12 *Let α be a nonzero element in K. Assume that both $R[\alpha]$ and $R[\alpha^{-1}]$ are flat over R. Then $R\langle\alpha\rangle = R$, i.e., α is anti-integral over R.*

Proof Take $P \in \mathrm{Spec}(R\langle\alpha\rangle)$ and put $p = P \cap R$. Then by Lemma 1.2.10, either $pR[\alpha] \neq R[\alpha]$ or $pR[\alpha^{-1}] \neq R[\alpha^{-1}]$ holds. We may assume $pR[\alpha] \neq R[\alpha]$ (by symmetry). Then there exists $Q \in \mathrm{Spec}(R[\alpha])$ such that $Q \cap R = p$. Since $R_p[\alpha]$ is flat over R_p, we have $R_p = (R_p[\alpha])_Q \supseteq R_p[\alpha] \supseteq R_p$. Thus $R_p[\alpha] = R_p$ and hence $(R\langle\alpha\rangle)_p = R\langle\alpha\rangle_P = R_p$, which implies that $R\langle\alpha\rangle$ is flat over R by Lemma 1.2.11. So $R\langle\alpha\rangle = R$ by [Ri] since $R\langle\alpha\rangle$ is integral over R. ∎

Corollary 1.2.13 *Let $\alpha = b/a \in K$ ($a, b \in R, ab \neq 0$). Then if both $R[\alpha]$ and $R[\alpha^{-1}]$ are flat over R, $(a, b)R$ is an invertible ideal of R.*

Proof The element α is anti-integral over R by Proposition 1.2.12. The conclusion follows from Theorem 1.2.8. ∎

Corollary 1.2.14 *Let $\alpha \in K$ be a nonzero element. Consider the following conditions:*

(1) *$R[\alpha]$ is an anti-integral extension of R;*

(2) *$R[\alpha]$ is flat over R;*

(3) *$R[\alpha^{-1}]$ is flat over R.*

Then two of these conditions imply the other.

Proof If (1) holds, (2) $\Leftrightarrow$ (3) follows from Proposition 1.2.9. (2) + (3) implies (1) by Proposition 1.2.12. $\square$

We recall that: Let A be an over-ring of R, which is of finite type over R. Assume that R is integrally closed in K and that A is quasi-finite over R (*cf.* [Ra]). Then the contraction map $\varphi : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$ is an open immersion, and if furthermore φ is surjective, then $A = R$. This is known as the Zariski Main Theorem (ZMT) (*cf.*[Ra]). We extend this in a certain sense (see Theorem 1.2.18).

We recall the following definitions.

Definition 1.2.15 ([R,(1.1.11)], [M2]). If T is an integral domain, then T satisfies the *altitude formula* or the *dimension formula* in case, for all finitely generated integral domain B over T and for all $P \in \mathrm{Spec}(B)$,

$$\mathrm{ht}(P) + \mathrm{Tr.deg}_{T/(P\cap T)} B/P = \mathrm{ht}(P \cap T) + \mathrm{Tr.deg}_T B$$

Proposition 1.2.16 *Let α be a nonzero element in K, which is anti-integral over R. Assume that R contains an infinite field k and that R satisfies the altitude formula. If the contraction map $\varphi : \mathrm{Spec}(R[\alpha]) \longrightarrow \mathrm{Spec}(R)$ is such that $\mathrm{Im}(\varphi) \supseteq \mathrm{Dp}_1(R)$, then $R[\alpha] = R$.*

Proof Take $p \in \mathrm{Spec}(R)$. If $I_\alpha \not\subseteq p$, then $R[\alpha] \subseteq R_p$, and so $R_p[\alpha] = R_p$. Suppose that $I_\alpha \subseteq p$ and that p is a prime divisor of I_α. Then $p \in \mathrm{Dp}_1(R)$. Since $\mathrm{Im}(\varphi) \supseteq \mathrm{Dp}_1(R)$, we have $\alpha I_\alpha \subseteq p$ by Theorem 1.2.6. Thus $\sqrt{I_\alpha} \supseteq \sqrt{\alpha I_\alpha}$. Note here that $I_\alpha = I_{\alpha-\mu}$ for any $\mu \in k$, that $R[\alpha] = R[\alpha-\mu]$, and that $\alpha-\mu$ is anti-integral over R by Lemma 1.1.9. We have $\sqrt{I_{\alpha-\mu}} \supseteq \sqrt{(\alpha-\mu)I_{\alpha-\mu}}$ for any $\mu \in k$. Now let q be in $\mathrm{Ht}_1(R)$ and suppose $q \supseteq I_\alpha$. Then $R[\alpha]$ is not a blowing-up at q over R. [Indeed, suppose that $R_q[\alpha]/qR_q[\alpha] \simeq (R_q/qR_q)[T]$, where T is an indeterminate. Then $qR_q[\alpha]$ is a prime ideal and $\mathrm{ht}(qR_q[\alpha]) = \mathrm{ht}(qR_q) + \mathrm{Tr.deg}_R R[\alpha] - \mathrm{Tr.deg}_{k(q)} R_q[\alpha]/qR_q[\alpha] = 1 + 0 - 1 = 0$ ($k(q) := R_q/qR_q$) by the altitude formula (*cf.* Definition 1.2.15 in Section 23), which implies that $qR_q[\alpha] = (0)$, a contradiction.] Hence β^{-1} is integral over R for $\beta = \alpha - \mu$ (some $\mu \in k$) by Lemma 1.2.2. Since β is anti-integral over R by Lemma 1.1.9,

$\beta^{-1} \in R_q$. Thus $\beta I_\beta = I_{\beta^{-1}} \not\subseteq q$, which contradicts with $q \supseteq \sqrt{I_\beta} \supseteq \sqrt{\beta I_\beta}$. Therefore $\mathrm{ht}(I_\alpha) > 1$. But since R satisfies the altitude formula, the element α is integral over R. [Indeed, putting $p = P \cap R$ for arbitrary $P \in \mathrm{Ht}_1(\overline{R})$, we have $\mathrm{ht}(p) = 1$ by the altitude formula. Thus $I_\alpha \not\subseteq p$ and hence $\alpha \in R_p \subseteq \overline{R}_p$, which yields that $\alpha \in \overline{R}$.] Therefore $R\langle\alpha\rangle = R[\alpha]$ by Lemma 1.1.3. So $R[\alpha] = R\langle\alpha\rangle = R$. ∎

As a corollary to the proof of Proposition 1.2.16, we have the following:

Corollary 1.2.17 *Let α be an anti-integral element in K. Assume that R contains an infinite field k and that $R[\alpha]$ is quasi-finite over R. If the contraction map $\varphi : \mathrm{Spec}(R[\alpha]) \longrightarrow \mathrm{Spec}(R)$ satisfies $\mathrm{Im}(\varphi) \supseteq \mathrm{Dp}_1(R)$, then $R[\alpha] = R$.*

Proof In the same way as in the proof of Proposition 1.2.16, we can show that $\sqrt{I_\alpha} \supseteq \sqrt{\alpha I_\alpha}$. Since $R[\alpha]$ is quasi-finite over R, $R[\alpha]$ is not a blowing-up at any $p \in \mathrm{Spec}(R)$ with $pR_p[\alpha] \neq R_p[\alpha]$. Hence by Theorem 1.2.6, $I_\alpha + \alpha I_\alpha \not\subseteq p$ for any $p \in \mathrm{Spec}(R)$ with $pR_p[\alpha] \neq R_p[\alpha]$. So $\sqrt{I_\alpha} \supseteq \sqrt{\alpha I_\alpha}$ implies that $I_\alpha \not\subseteq p$ for any $p \in \mathrm{Spec}(R)$ with $pR_p[\alpha] \neq R_p[\alpha]$. Thus $\alpha \in R_p$ for any $p \in \mathrm{Spec}(R)$ with $pR_p[\alpha] \neq R_p[\alpha]$. But since $\mathrm{Im}(\varphi) \supseteq \mathrm{Dp}_1(R)$, we have $pR[\alpha] \neq R[\alpha]$ for any $p \in \mathrm{Dp}_1(R)$, and hence $R = \bigcap R_p \supseteq \bigcap R_p[\alpha] \supseteq R[\alpha]$ $(p \in \mathrm{Dp}_1(R))$, which implies that $R = R[\alpha]$. ∎

Theorem 1.2.18 *Assume that R contains an infinite field k and that R satisfies the altitude formula. Let $A = R[\alpha_1, \ldots, \alpha_n]$ with $\alpha_i \in K$ anti-integral over R. If the contraction map $\varphi : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$ satisfies $\mathrm{Im}(\varphi) \supseteq \mathrm{Dp}_1(R)$, then $A = R$.*

Proof From the commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(R[\alpha_i]) \\ \varphi\big\downarrow & & \varphi_i\big\downarrow \\ \mathrm{Spec}(R) & = & \mathrm{Spec}(R) \end{array}$$

we have $\mathrm{Im}(\varphi_i) \supseteq \mathrm{Im}\,\varphi \supseteq \mathrm{Dp}_1(R)$. Thus $\alpha_i \in R$ for each i by Proposition 1.2.16. Hence $A = R$. ∎

Corollary 1.2.19 *Under the same assumption as in Theorem 1.2.18, The following statements are equivalent:*

(1) *φ is a surjection;*

(2) *A is integral over R;*

(3) *$A = R$.*

Corollary 1.2.20 *Let $A = R[\alpha_1, \ldots, \alpha_n]$ with nonzero element $\alpha_i \in K$ anti-integral over R. Assume that R contains an infinite field k and that A is quasi-finite over R. If the contraction map $\varphi : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$ satisfies $\mathrm{Im}(\varphi) \supseteq \mathrm{Dp}_1(R)$, then $A = R$.*

Remark 1.2.21 Let A and R be the same as in Theorem 1.2.18. If A is integral over R, $\varphi : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$ is obviously surjective. But the surjectivity of φ does not necessarily ensure the integrality of A over R. For example: $A = k[T, 1/(T-1)] \supseteq k[T] \supseteq R = k[T(T-1), T^2(T-1)]$, where k is a field. Theorem 1.2.18 asserts that if A is anti-integral over R, the surjectivity of φ ensures the integrality of A over R. ∎

1.3 The Ring $\mathcal{R}(I_\alpha)$ and the Anti-Integrality of α

To begin with, we introduce the following over-ring of R.

Definition 1.3.1 Let J be a fractional ideal of R. Define

$$\mathcal{R}(J) := \{\alpha \in K \,|\, \alpha J \subseteq J\}$$

It is easy to see that $\mathcal{R}(J)$ is an over-ring of R, which is integral over R, and $\mathcal{R}(J_p) = \mathcal{R}(J)_p$ for any $p \in \mathrm{Spec}(R)$.

Proposition 1.3.2 *Let J be an integral ideal of R. Then the following statements are equivalent:*

(1) $\mathcal{R}(J^{-1}) = R$ *(where $J^{-1} := R :_K J = \mathrm{Hom}_R(J, R)$);*

(2) *J_p is a principal ideal of R_p for each $p \in \mathrm{Dp}_1(R)$.*

Proof (2) $\Rightarrow$ (1): Since $R = \cap R_p$ ($p \in \mathrm{Dp}_1(R)$), it suffices to show that $\mathcal{R}(J^{-1}) \subseteq R_p$ for any $p \in \mathrm{Dp}_1(R)$. As $J_p = \alpha R_p$ for some $\alpha \in R$, we have $J_p^{-1} = \alpha^{-1} R_p$. Hence $R_p \subseteq \mathcal{R}(J_p^{-1}) = \mathcal{R}(\alpha^{-1} R_p) \subseteq R_p$. The containment $\mathcal{R}(J^{-1}) \subseteq \mathcal{R}(J_p^{-1})$ is obvious. Thus $\mathcal{R}(J^{-1}) \subseteq \mathcal{R}(J_p^{-1}) \subseteq R_p$, as was wanted. (1) $\Rightarrow$ (2): We may assume that (R, m) is a local domain and that $m \in \mathrm{Dp}_1(R)$. If R is a discrete valuation ring, R is a principal ideal domain. Hence J is a principal ideal of R. Suppose that R is not a discrete valuation ring (DVR) of rank one. Then $m :_K m \neq R$. If there exists $\phi \in J^{-1} = \mathrm{Hom}_R(J, R)$ with

$\phi(J) = R$, an exact sequence: $J \xrightarrow{\phi} R \longrightarrow 0$ splits. Since $\operatorname{rank} J = 1$, we have $J \simeq R$. So we suppose that $\phi(J) \neq R$ for any $\phi \in J^{-1}$. In this case, $\phi(J) \subseteq m$. We have $\alpha\phi(J) \subseteq \alpha m \subseteq m \subseteq R$ for any $\alpha \in m :_K m$, and hence $\alpha \in \mathcal{R}(J^{-1})$. Thus $m :_K m \subseteq \mathcal{R}(J^{-1})$. But since $\mathcal{R}(J^{-1}) \supseteq m :_k m \underset{\neq}{\supset} R$, we have $R = \mathcal{R}(J^{-1}) \underset{\neq}{\supset} R$, which is a contradiction. $\square$

Proposition 1.3.3 *Let J be a divisorial integral ideal of R. If $\mathcal{R}(J^{-1}) = R$, then there exists a nonzero element $\alpha \in K$ such that $J = I_\alpha$.*

Proof We first show the following claim: If $p \in \mathrm{Dp}_1(R)$ contains J, then p is a prime divisor of J.
In fact, since $\mathcal{R}(J^{-1}) = R$, it follows that $J_p = aR_p$ for some $a \in J$ by Proposition 1.3.2. As pR_p is a prime divisor of a principal ideal $aR_p = J_p$ (*cf.* [Y]), p is a prime divisor of J. Let $J = q_1 \cap \cdots \cap q_n$ ($\sqrt{q_i} = p_i$) be a primary decomposition. Then $p_i \in \mathrm{Dp}_1(R)$ because J is divisorial. Put $S = R \setminus \bigcup p_i$. Then $A := S^{-1}R$ is semi-local and each maximal ideal of A is of depth 1. Since JA is invertible by the above claim and since A is semi-local, JA is a principal ideal. So there exists $a \in J$ such that $JA = aA$. Hence we have a primary decomposition:

$$aR = q_1 \cap \cdots \cap q_n \cap Q_1 \cap \cdots \cap Q_\ell \ \ (\sqrt{Q_i} = P_i,\ P_i \neq p_1, \ldots, p_n)$$

Suppose that $P_i \supseteq J$ for some i. Then $P_i \in \mathrm{Dp}_1(R)$ implies that P_i is a prime divisor of J by the first claim. But since the prime divisors of J are $p_1, \ldots, p_n$, this is absurd. Hence $J \not\subseteq P_i$ for all $i = 1, \ldots, m$. Next we will show that $P_i \not\subseteq p_j$ for any i, j. Suppose that $P_i \subseteq p_j$. Then $JR_{p_j} = aR_{p_j}$, and $a \in P_i$ implies $JR_{p_j} \subseteq P_iR_{p_j}$. Thus $J \subseteq P_i$, a contradiction. Hence we have

$$Q_1 \cap \cdots \cap Q_m \not\subseteq p_1, \ldots, p_n$$

So there exists $b \in Q_1 \cap \cdots \cap Q_m$, $b \notin p_1, \ldots, p_n$. We have $(aR :_R bR) = q_1 \cap \cdots \cap q_n = J$. Putting $\alpha = b/a \in K$, we have $I_\alpha = aR :_R bR = J$. $\square$

Theorem 1.3.4 *Let α be a nonzero element in K. If $\mathcal{R}(I_\alpha) = R$, then $R\langle\alpha\rangle = R$, i.e., α is anti-integral over R.*

Proof *Claim* 1: For each $p \in \mathrm{Dp}_1(R)$, either $(I_\alpha)_p = R_p$ or $(I_{\alpha^{-1}})_p = R_p$ holds. Indeed, let $p \in \mathrm{Dp}_1(R)$. If $I_\alpha \not\subseteq p$, then $(I_\alpha)_p = R_p$. Suppose that $I_\alpha \subseteq p$ and $I_{\alpha^{-1}} \subseteq p$. Then $(I_\alpha)_p = aR_p$ for some $a \in I_\alpha$ by Proposition 1.3.2. So $\alpha aR_p = \alpha(I_\alpha)_p = (I_{\alpha^{-1}})_p \subseteq pR_p$. Thus $\alpha a = r$ for some $r \in pR_p$

and $\alpha = r/a$. So $aR_p = (I_\alpha)_p = (aR_p :_{R_p} rR_p)$, which asserts that a, r is an R_p-regular sequence. But depth$(R_p) = 1$, a contradiction. Therefore either $(I_\alpha)_p = R_p$ or $(I_{\alpha^{-1}})_p = R_p$ holds.
Claim 2: If $\mathcal{R}(I\alpha) = R$, then $I_\alpha + I_{\alpha^{-1}} \not\subseteq p$ for each $p \in \mathrm{Dp}_1(R)$.
In fact, suppose that $I_\alpha + I_{\alpha^{-1}} \subseteq p$ with $p \in \mathrm{Dp}_1(R)$. Then both $(I_\alpha)_p$ and $(I_{\alpha^{-1}})_p$ are principal ideals in R_p by Proposition 1.3.2. But by Claim 1, either $(I_\alpha)_p = R_p$ or $(I_{\alpha^{-1}})_p = R_p$ holds. This contradicts with $(I_\alpha)_p + (I_{\alpha^{-1}})_p \subseteq pR_p$. Thus for $p \in \mathrm{Dp}_1(R)$, $I_\alpha + I_{\alpha^{-1}} \not\subseteq p$ by Claim 2. Hence $\alpha \in R_p$ or $\alpha^{-1} \in R_p$. Therefore $R\langle\alpha\rangle = R[\alpha] \cap R[\alpha^{-1}] \subseteq R_p$ for all $p \in \mathrm{Dp}_1(R)$. So $R\langle\alpha\rangle = R$ by Lemma 1.1.5. $\square$

Corollary 1.3.5 *Let J be an invertible ideal contained in R. Then there exists an anti-integral element α of K such that $J = I_\alpha$.*

Proof Since $\mathcal{R}(J^{-1}) = R$ by Proposition 1.3.2, $J = I_\alpha$ for some nonzero element α in K by Proposition 1.3.2. Hence $R\langle\alpha\rangle = R$ by Theorem 1.3.4. $\square$

Remark 1.3.6 The converse of Theorem 1.3.4 does not necessarily hold as can be seen in the following example:

Let (R, m) be a Noetherian local domain such that $Ass_R(\overline{R}/R) = \{m\}$, ht$(m) > 1$. Take $\alpha \in K$ such that $I_\alpha = q_1 \cap \ldots \cap q_t$, where q_i is a primary ideal with ht$(q_i) = 1$ for each i. Then $\sqrt{q_i} \notin \mathrm{Ass}_R(\overline{R}/R)$ implies that $R\langle\alpha\rangle = R$ by Corollary 1.1.11. But since depth$(R) = 1$, I_α cannot be a principal ideal, and hence $\mathcal{R}(I_\alpha) \neq R$ (*cf.* Proposition 1.3.2). $\square$

1.4 Strictly Closedness and Integral Extensions

We consider integral extensions. Let R be a Noetherian domain with quotient field K. Assume that the integral closure $\overline{R}$ of R in K is a finite R-module. We recall the following definition introduced by Lipman ([L]).

Definition 1.4.1 Let A be an over-ring of R and ${}^*_A R$ be the subring of A defined as

$$ {}^*_A R := \{\alpha \in A \mid \alpha \otimes 1 = 1 \otimes \alpha \in A \otimes_R A\} $$

We say that R is *strictly closed* in A if ${}^*_A R = R$.

An intermediate ring A between R and $\overline{R}$ is called a *birational-integral extension* of R. For the strictly closedness of a birational-integral extension of R, the following proposition is important.

Proposition 1.4.2 ([KY5, Proposition 2]) *Let A be a birational-integral extension of R. If R is strictly closed in A, then $I_\alpha \cap I_\beta \subseteq I_{\alpha\beta}$ for any elements $\alpha, \beta \in A$.*

We show the following proposition, which is a characterization of birational-integral extensions in terms of denominator ideals. Note that $I_{\alpha^{-1}} = \alpha I_\alpha$.

Theorem 1.4.3 *Let $A = R[\alpha]$ with $\alpha \in K$. Suppose that R is strictly closed in A. Then α is integral if and only if $I_{\alpha^{-1}} \subseteq I_\alpha$.*

Proof ($\Rightarrow$): By the determinant trick, this implication is clear.
($\Leftarrow$): Let C be the conductor of A over R. Then it is easy to see that $C = \bigcap_{i=1}^{n} I_{\alpha^i}$, for some n. Since R is strictly closed in A by assumption, we have $I_\alpha \cap I_\beta \subseteq I_{\alpha\beta}$ for any elements by Proposition 1.4.2. Hence it holds that $I_\alpha \subseteq I_{\alpha^i}$ for all i, and $I_\alpha \subseteq C$. Thus $C = I_\alpha$. Since the conductor C is an ideal of R, I_α is an ideal in A, and $\alpha I_\alpha \subseteq I_\alpha$. $\square$

1.5 Upper-Prime, Upper-Primary, or Upper-Quasi-Primary Ideals

Let R be a Noetherian domain and K its quotient field. Let α be a nonzero element in K. We call α anti-integral over R if $R = R[\alpha] \cap R[\alpha^{-1}]$ and, in this case, a simple birational-extension $R[\alpha]$ is called an anti-integral extension of R (see Section 1). For $\alpha \in K$, I_α denotes an ideal $R :_R \alpha = \{a \in R | a\alpha \in R\}$ of R. A. Mirbagheri and L. J. Ratliff, Jr. proved that if $R[\alpha]$ is an anti-integral extension of R, then $\mathrm{Ker}(\pi)$ is generated by linearly related polynomials of α (that is, $aX - b$, $\alpha = b/a$), where $\pi : R[X] \longrightarrow R[\alpha]$ is the natural R-algebra homomorphism. Thus if $p \supseteq I_\alpha + \alpha I_\alpha$ for $p \in \mathrm{Spec}(R)$, then it follows that $R[\alpha]/pR[\alpha] \cong (R/p)[T]$, where T is an indeterminate (Theorem 1.2.6). Anti-integral extensions were studied before.

We list some known results (Theorem 1.2.8), which will be used frequently: Let $\alpha \in K$ be a nonzero element. Then

(1) If $I_\alpha + \alpha I_\alpha = R$, then $R[\alpha]$ is flat over R.

(2) Assume that $\alpha = b/a$ $(a, b \in R)$ is anti-integral over R. Then the following statements are equivalent:

(2.i) $R[\alpha]$ is flat over R;

(2.ii) $I_\alpha + \alpha I_\alpha = R$;

(2.iii) $(a, b)R$ is an invertible ideal of R.

If for $p \in \mathrm{Spec}(R)$, $p \not\supseteq I_\alpha + \alpha I_\alpha$, then $R_p[\alpha] = (R[\alpha])_p$ is flat over R_p by (1).

Let S be a birational extension of R.

Definition 1.5.1 (*cf.* [Mc]) Let q be an ideal of R. We say that q is an *upper-prime* ideal of R with respect to S if $a\alpha \in qS$ for $a \in R, \alpha \in S$ implies $a \in q$ or $\alpha \in qS$.

The following results are proved in [S1].

Remark 1.5.2

(1) Let I be an ideal of R such that $IS \neq S$. If I is an upper-prime ideal of R with respect to S, then $IS \cap R = I$ and I is a prime ideal of R.

(2) Let $p \in \mathrm{Spec}(R)$. Then p is an upper-prime ideal of R with respect to S if and only if $pS = S$ or $pS_p \cap S = pS$.

(3) Let $p \in \mathrm{Spec}(R)$. Then p is an upper-prime ideal of R with respect to S if and only if either $pS = S$ or $pS \neq S$, and $P \cap R = p$ for any prime divisor P of the ideal pS. $\square$

Now, we give the following theorem.

Theorem 1.5.3 *Let R be a Noetherian domain and K its quotient field. Let α be an anti-integral element in K. Put $S = R[\alpha]$, $I_\alpha := \{a \in R \mid a\alpha \in R\}$ and $J_\alpha := I_\alpha + \alpha I_\alpha$. For $p \in \mathrm{Spec}(R)$, the following statements hold:*

(i) *assume that $p \supseteq J_\alpha$. Then $pS \in \mathrm{Spec}(S)$ and $pS \cap R = p$;*
Consequently p is an upper-prime ideal of R with respect to S. Moreover, $S/pS = (R/p)[T]$ holds, where T is an indeterminate.

(ii) *assume that $J_\alpha \not\subseteq p$ and $p + J_\alpha \neq R$;*

(ii.a) *if $p \supseteq I_\alpha$ and $p \not\supseteq \alpha I_\alpha$, then $pS_p = S_p$. Also, the converse is valid. Hence it follows that $pS \cap R \underset{\neq}{\supset} p$ and p is not an upper-prime ideal of R with respect to S;*

(ii.b) *if αI_α, then p is not an upper-prime ideal of R with respect to S. Thus there exists some primary components Q_i of pS such that*

$$pS = (pS_p \cap S) \cap Q_i \cap \cdots \cap Q_l$$

and

$$\sqrt{Q_i} \cap R \underset{\neq}{\supset} p$$

(iii) *assume that* $p + J_\alpha = R$. *Then* p *is an upper-prime ideal of* R *with respect to* S. *In particular, it follows that* $pS \in \mathrm{Spec}(S)$, $pS \cap R = p$ *and* $S/pS = R/p$.

Proof (i) Let $\pi : R[X] \longrightarrow R[\alpha]$ be the canonical ring homomorphism. Then $\mathrm{Ker}(\pi)$ is generated by linear related polynomials of α. And J_α is the ideal of R generated by the coefficients of the elements of $\mathrm{Ker}(\pi)$. Since $p \supseteq J_\alpha$ by assumption, we have $pR[X] \subseteq \mathrm{Ker}(\pi)$. Hence $R[\alpha]/pR[\alpha] \cong (R/p)[T]$. Consequently pS is a prime ideal of S and $pS \cap R = p$. Thus p is an upper-prime ideal of R with respect to S.
(ii.a) There exist two elements a and b of R such that $a \in I_\alpha \subseteq p$ and $b = a\alpha \notin p$ by assumption. Then $b = a(b/a) \in pS_p$. Thus we have $pS_p = S_p$. Hence p is not an upper-prime ideal of R with respect to S. Conversely, assume that $pS_p = S_p$. Since (i) does not hold in this case, it holds that $p \not\supseteq J_\alpha$. Suppose that $p \not\supseteq I_\alpha$, we would have that $S_p = R_p$ and so $pS_p \neq S_p$. This contradicts the assumption. Hence $p \supseteq I_\alpha$. Since $p \supseteq J_\alpha$, it follows that $p \supseteq \alpha I_\alpha$.
(ii.b) By assumption, there exists $a, b \in R$ such that $\alpha = b/a$ with $a \in I_\alpha \setminus p$. Then $a\alpha = b \in \alpha I_\alpha \subseteq p \subseteq pS$. Suppose that p is an upper-prime ideal of R with respect to S. Since $a \notin p$ and $I_\alpha \not\subseteq p$, it holds that $\alpha \in pS$. Hence $S/pS = R/p$. Since $p + J_\alpha \neq R$, there is a maximal ideal m of R such that $p + J_\alpha \subseteq m$. It is obvious that $\alpha \in pS \subseteq mS$ and so $S/mS = R/m$. But, since $J_\alpha \subseteq m$, we have $S/mS = R/m[T]$ by (ii,a) (replacing p by m), which is a contradiction. Thus p is not an upper-prime ideal of R with respect to S. There exist some primary components Q_i such that

$$pS = (pS_p \cap S) \cap Q_1 \cap \cdots \cap Q_l$$

where $\sqrt{Q_i} \cap R \underset{\neq}{\supset} p$.
(iii) Suppose that p is not an upper-prime ideal of R with respect to S. Then there exist some primary components Q_i such that a primary decomposition is

$$pS = (pS_p \cap S) \cap Q_1 \cap \cdots \cap Q_l$$

where $\sqrt{Q_i} \cap R\ (= q_i) \underset{\neq}{\supset} p$. Since $p + J_\alpha = R$ and $q_i \underset{\neq}{\supset} p$, we have that $q_i = q \not\supseteq J_\alpha$. Since S_q is a flat extension of $/R_q$, it holds that pR_q is an upper-prime ideal of R with respect to S. There exists some element $a \in I_\alpha \setminus p$. Since $b = a\alpha \in \alpha I_\alpha \subseteq p \subseteq pS$, it follows that $\alpha \in pS$. Hence $S/pS = R/p$. Thus pS is a prime ideal of S and $pS \cap R = p$. $\square$

Remark 1.5.4 If $p \not\supseteq J_\alpha$ and $p \supseteq \alpha I_\alpha$, then we have $p \not\supseteq I_\alpha$. Therefore there exists an element $a \in I_\alpha \setminus p$. Put $b = a\alpha$. Then $b \in \alpha I_\alpha \subseteq p$. This is the case (iii) in Theorem 1.5.3. So another case is that $p \not\supseteq I_\alpha$ and $p \not\supseteq \alpha I_\alpha$. In this case, it is not known whether p is an upper-prime ideal of R with respect to S or not. In the case (ii), if $pS \neq S$, it follows that $pS = qS$ and $S/pS = (R/q)[T]$, where $q = pS \cap R$ and T is an indeterminate. It is not known whether q is prime or not. In the case (iii), the minimal prime divisors of $p + J_\alpha$ appear among $\sqrt{Q_1} \cap R, \ldots, \sqrt{Q_l} \cap R$. $\square$

Example 1.5.5 Let k be a field and $R = k[x, y]$ be a polynomial ring. Put $\alpha = y/x$, $p = xR$ and $S = R[y/x]$. Then $R \subseteq S$ and $y \in xS \cap R$. Hence $xS \cap R \underset{\neq}{\supset} p$ and $pS_p = S_p$. This is the case that $p \supseteq I_\alpha$ and $p \not\supseteq J_\alpha$.

Example 1.5.6 R, k, S, α are the same as in Example 1.5.5. Put $p = yR$. Then we have that $p \not\supseteq J_\alpha$, $p \not\supseteq I_\alpha$, $p \supseteq \alpha I_\alpha$, $pS_p \cap R = p$ and $pS = p$. But p is not an upper-prime ideal of R with respect to S. It holds that $pS = (y/x)S \cap xS$, $(y/x)S \cap R = yR$ and $xS = (x, y)R$. The same results hold for $p = (x + y)R$.

Example 1.5.7 Let k be a field and $R = k[x, y, z]$ be a polynomial ring in three variables over k. Put $\alpha = y/x$ and $p = zR$. Then pS is a prime ideal of S and $pS \cap R = p$. Hence p is an upper-prime ideal of R with respect to S. This example is applied to the case that $p \not\supseteq I_\alpha$, $p \not\supseteq \alpha I_\alpha$. Consequently it follows that $p \not\supseteq J_\alpha$.

For a primary decomposition of pS, the following result holds.

Proposition 1.5.8 *Let $p \in \mathrm{Spec}(R)$. Assume that $pS_p \neq S_p$ and p is not an upper-prime ideal of R with respect to S. Then pS has a primary decomposition:*

$$pS = (pS_p \cap S) \cap Q_1 \cap \cdots \cap Q_l$$

such that $q_i \supseteq p + J_\alpha$, where $q_i = \sqrt{Q_i} \cap R$.

Proof Since $p \subseteq pS \subseteq Q_i$, it is clear that $p \subseteq q_i$. Assume that $q_i \not\supseteq J_\alpha$. We have that S_{q_i}/R_{q_i} is a flat extension by Theorem 1.2.8 and $pS_{q_i} \neq S_{q_i}$. It follows that $\mathrm{Ass}_R(S_{q_i}/pS_{q_i}) = \{pR_{q_i}\}$, but $q_i R_{q_i} \in \mathrm{Ass}_R(S_{q_i}/pS_{q_i})$. This is a contradiction. $\square$

Let R be a Noetherian domain and K its quotient field and let $S \supseteq R$ be a birational extension of integral domains. For $\alpha \in K$, I_α denotes an ideal $R :_R \alpha = \{a \in R | a\alpha \in R\}$ of R. Put $J_\alpha = I_\alpha + \alpha I_\alpha$.

Definition 1.5.9 Let q be an ideal of R. We say that q is an *upper-primary* ideal of R with respect to S if $a\alpha \in qS$ for $a \in R, \alpha \in S$ implies $a \in q$ or $\alpha \in \sqrt{qS}$, and that q is an *upper-quasi-primary* ideal of R with respect to S if $a\alpha \in qS$ for $a \in R, \alpha \in S$ implies $a \in \sqrt{q}$ or $\alpha \in qS$.

Example 1.5.10 Let $\mathbf{Z}$ denote the ring of rational integers. Put $R = \mathbf{Z}$ and $S = \mathbf{Z}[1/2]$. Then every primary ideal of R is an upper-primary ideal of R with respect to S. Moreover, the upper-primary ideal of R with respect to S ideals of R coincide with the upper-quasi-primary ideals of R with respect to S. Also, if q is an ideal of R such that $qS \neq S$, then q is an upper-primary ideal of R with respect to S if and only if q is a primary ideal. Then principal ideal $2aR$ such that a is not 2^n(n is any nonnegative integer) is a an upper-primary ideal of R with respect to S ideal of R, but not a primary ideal.

Remark 1.5.11

(1) Let q be a prime ideal of R. If q is an upper-prime ideal of R with respect to S, then q is an upper-primary ideal of R with respect to S.

(2) Let q be an ideal of R such that $qS \neq S$. If q is an upper-primary ideal of R with respect to S, then $qS \cap R = q$ and q is a primary ideal of R.

(3) Let q be an ideal of R and let

$$qS = Q_1 \cap \cdots \cap Q_n \cap T_1 \cap \cdots \cap T_m$$

denote an irredundant primary decomposition of the ideal qS, where the Q_i are the isolated primary components and the T_j are the embedded primary components. Then q is an upper-primary ideal of R with respect to S if and only if $Q_i \cap R = q$ for all Q_i.

$\square$

Let α be an anti-integral element in K and put $S := R[\alpha]$. We consider the upper-primary ideals of R with respect to S.

Example 1.5.12 Let k be a field and $R = k[x, y]$ be the polynomial ring in two indeterminates over k. Put $\alpha = y/x$, $S = R[\alpha]$ and $q = (x, y)^2R = (x^2, xy, y^2)R$. Then we have that $qS = (x^2, x^2\alpha, x^2\alpha^2)S = x^2S$ and qS is a primary ideal of S. Since $x^2S \cap R = q$, it follows that q is an upper-primary ideal of R with respect to S by Remark 1.5.11. Put $q' = (x^2, y^2)R$. Since $q'S = x^2S$ and $q'S \cap R = q \neq q'$, we see that q' is not an upper-primary ideal of R with respect to S but both q and q' are primary ideals belonging to $(x, y)R$.

The following are the properties of a primary ideal which is neither an upper-primary ideal of R with respect to S nor an upper-quasi-primary ideal of R with respect to S.

Proposition 1.5.13 *Let $S = R[\alpha]$ and α be an anti-integral element in K. Let p be a prime ideal of R. Assume that $\alpha = b/a(a, b \in R, a \notin p, b \in p)$ and $p + J_\alpha \neq R$. If q is a p-primary ideal, then q is neither an upper-primary ideal of R with respect to S nor an upper-quasi-primary ideal of R with respect to S.*

Proof Suppose that q is p-primary. Since q is a an upper-primary ideal of R with respect to S, there exists a positive integer n such that $b^n = a^n\alpha^n \in q \subseteq qS$. Hence $\alpha^n \in \sqrt{qS}$. Now, since $p + J_\alpha \neq R$, there exists a maximal ideal m of R such that $p + J_\alpha \subseteq m$. Thus $S/mS \cong (R/m)[T]$, where T is an indeterminate. Therefore mS is a prime ideal of S. Since $\alpha^n \in \sqrt{qS} \subseteq \sqrt{pS} \subseteq mS$, it follows that $\alpha \in mS$. Hence $S/mS = R/m$, it contradicts. The later part of Proposition 1.5.13 is clear from the above proof. ∎

1.6 Some Subsets of Spec(R) in the Birational Case

Let R be a Noetherian integral domain with quotient field K, and let α be a nonzero element of K.

Write $A_1 := R[\alpha]$ $(=: A)$, $A_2 := R[\alpha^{-1}]$, $I_\alpha := \{r \in R \mid r\alpha \in R\}$, and $J_\alpha = I_\alpha + \alpha I_\alpha$. We recall that α is an anti-integral element or that α is anti-integral over R if $A_1 \cap A_2 = R$. When α is anti-integral over R, $R[\alpha]$ is said to be an anti-integral extension of R.

Let $\varphi_{A_i/R} : \mathrm{Spec}(A_i) \longrightarrow \mathrm{Spec}(R)$ be maps $(i = 1, 2)$ which satisfy the condition $\varphi_{A_i/R}(P) = P \cap R$ for any $P \in \mathrm{Spec}(A_i)$. For any ideal N of R, we define

$$V(N) = \{P \in \mathrm{Spec}(R) \mid P \supseteq N\}$$

We write $D(N)$ to denote the set $\mathrm{Spec}(R) \setminus V(N)$. Let $\Delta_{A/R} := \{p \in \mathrm{Spec}(R) | pA = A\}$ and $\Gamma_N := \{p \in \mathrm{Spec}(R) | p + N = R\}$.

Theorem 1.6.1 *Assume that α is anti-integral over R. The following results hold:*

(1) $\Delta_{A/R} = V(I_\alpha) \cap \Gamma_{J_\alpha}$;

(2) $\mathrm{Im}(\varphi_{A/R}) = D(I_\alpha) \cup V(J_\alpha)$, *where $\varphi = \varphi_{A/R} : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is a map such that $\varphi_{A/R}(P) = P \cap R$ for any $P \in \mathrm{Spec}(A)$;*

(3) $\Delta_{A_1/R} \cap \Delta_{A_2/R} = \emptyset$;

(4) $\mathrm{Im}(\varphi_{A_1/R}) \cup \mathrm{Im}(\varphi_{A_2/R}) = \mathrm{Spec}(R)$.

Proof (2) Suppose that $p \in D(I_\alpha) \cup V(J_\alpha)$. When $p \in V(J_\alpha)$, we claim that $p \in \mathrm{Im}(\varphi)$. Indeed, we have that $A/pA \cong (R/p)[T]$ by Theorem 1.2.6, where T is an indeterminate. Thus pA is a prime ideal of A and $pA \cap R = p$. Hence we have $p \in \mathrm{Im}(\varphi)$. In the case that $p \in D(I_\alpha)$, it follows that $A_p = R_p$. Write $pR_p \cap A = P$. Then $P \in \mathrm{Spec}(A)$ and $P \cap R = p$. So it follows that $p \in \mathrm{Im}(\varphi)$. Hence we have proved that $D(I_\alpha) \cup V(J_\alpha) \subseteq \mathrm{Im}(\varphi)$. To prove the opposite inclusion, assume that $p \notin D(I_\alpha) \cup V(J_\alpha)$. We claim that $p \notin \mathrm{Im}(\varphi)$. Suppose that $p \in \mathrm{Im}(\varphi)$. Then there exists a prime ideal P of A such that $P \cap R = p$, since $p \supseteq I_\alpha$. $\square$

Chapter 2

Simple Extensions of High Degree

Let R be a Noetherian integral domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic field extension L of the quotient field K of R and let $\pi : R[X] \longrightarrow R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Let $I_{[\alpha]} := \bigcap_{i=1}^{d}(R :_R \eta_i)$. For $f(X) \in R[X]$, let $\mathsf{c}(f(X))$ denote the ideal generated by the coefficients of $f(X)$. Let $J_{[\alpha]} := I_{[\alpha]}\mathsf{c}(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called an anti-integral element of degree d over R if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, $R[\alpha]$ is called an anti-integral extension of R. In the case $K(\alpha) = K$, an anti-integral element α is the same as an anti-integral element (i.e., $R = R[\alpha] \cap R[\alpha^{-1}]$) defined in Definition 1.1.4. The element α is called a super-primitive element of degree d over R if $J_{[\alpha]} \not\subseteq p$ for all primes p of depth 1, i.e., $p \in \mathrm{Dp}_1(R)$.

For $p \in \mathrm{Spec}(R)$, $k(p)$ denotes the residue field R_p/pR_p and $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p)$ denotes the dimension as a vector space over $k(p)$. In this chapter, we are mainly interested in characterizing the flatness and the integrality of an anti-integral extension $R[\alpha]$ of R. Among others we obtain the following results:

$R[\alpha]$ is flat over R if and only if $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p) \leq d$ for all $p \in \mathrm{Spec}(R)$.

Concerning a super-primitive element, we obtain that if R is a Krull domain and α is an algebraic element over R, then α is a super-primitive element. We also obtain that a super-primitive element is an anti-integral element. More precisely, α is super-primitive over R if and only if α is anti-integral over R and $R[\alpha]_p$ is flat over R_p for any prime ideal p of depth 1.

Notations and Conventions

Throughout this chapter, we use the following notations unless otherwise specified:

R: a Noetherian integral domain,

$K = K(R)$: the quotient field of R,

L: an algebraic field extension of K,

α: a nonzero element of L,

$d = [K(\alpha) : K]$, and

$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$, the minimal polynomial of α over K.

Let $\pi : R[X] \longrightarrow R[\alpha]$ be an R-algebra homomorphism defined by $X \longrightarrow \alpha$ and let $E^{(\alpha)} := \mathrm{Ker}(\pi)$. Then $E^{(\alpha)}$ is a prime ideal of $R[X]$ with $E^{(\alpha)} \cap R = (0)$. By definition, $E^{(\alpha)} = \{\psi(X) \in R[X] | \psi(\alpha) = 0\}$.

Let $I_{[\alpha]} := \bigcap_{i=1}^{d}(R :_R \eta_i)$, which is an ideal of R. For $f(X) \in K[X]$, $\mathsf{c}(f(X)) :=$ the ideal generated by all coefficients of $f(X)$, that is, $\mathsf{c}(f(X))$ is the *content ideal* of $f(X)$. Let $J_{[\alpha]} := I_{[\alpha]}\mathsf{c}(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$ and let $\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})$.

We also use the standard notations as follows:

$k(p)$:=the residue field R_p/pR_p for $p \in \mathrm{Spec}(R)$,

$\mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) \mid \mathrm{depth}(R_p) = 1\}$, and

$\mathrm{Ht}_1(R) := \{p \in \mathrm{Spec}(R) \mid \mathrm{ht}(p) = 1\}$.

2.1 Sharma Polynomials

We begin with the following definition.

Definition 2.1.1 Let $f(X) = a_0X^n + a_1X^{n-1} + \cdots + a_n$ be a polynomial in $R[X]$. We say that $f(X)$ is a *Sharma polynomial* in $R[X]$ if there does not exist $t \in R$ with $t \notin a_0R$ such that $ta_i \in a_0R$ for $1 \leq i \leq n$.

We give an equivalent condition for a polynomial to be a Sharma polynomial in the following proposition.

Theorem 2.1.2 *Let $f(X)$ be a polynomial in $R[X]$. Then $f(X)$ is a Sharma polynomial if and only if $\mathsf{c}(f(X)) \nsubseteq p$ for any $p \in Dp_1(R)$.*

Proof Let $f(X) = a_0X^n + \cdots + a_n$ ($a_i \in R$).
($\Rightarrow$) Suppose that $\mathsf{c}(f(X)) \subseteq p$ for some $p \in \mathrm{Dp}_1(R)$. Then $a_0 \in p$, and there exists $t \notin a_0R$ such that $p = (a_0R :_R t)$. In this case, $a_i \in p$ implies that $a_it \in a_0R$ ($1 \leq i \leq n$), which asserts that $f(X)$ is not a Sharma polynomial.

($\Leftarrow$) Suppose that $f(X)$ is not a Sharma polynomial. Then there exists $t \in R$ such that $t \notin a_0R$, $ta_i \in a_0R$ $(1 \leq i \leq n)$. Since there exists $p \in \mathrm{Dp}_1(R)$ such that $(a_0R :_R t) \subseteq p$, we have $a_i \in (a_0R :_R t) \subseteq p$ $(1 \leq i \leq n)$ and obviously $a_0 \in p$. So $\mathsf{c}(f(X)) = (a_0, \ldots, a_n) \subseteq p$, a contradiction. $\square$

Proposition 2.1.3 *Let $R[X]$ be a polynomial ring over R and let P be a nonzero prime ideal in $R[X]$ such that $P \cap R = (0)$. Let $f(X) \in P$ be a polynomial of least positive degree. Then $P = f(X)R[X]$ if $f(X)$ is a Sharma polynomial.*

Proof Assume $P = f(X)R[X]$ and put $f(X) = a_0X^d + \cdots + a_d$. Suppose there exists $t \notin (a_0)$ such that $ta_i \in (a_0)$ for $1 \leq i \leq d$. We will arrive at a contradiction. Let $ta_i = b_ia_0$ with $b_i \in R$ $(i = 1, \ldots, d)$. Then $a_0(tX^d + b_1X^{d-1} + \cdots + b_d) = tf(X) \in P$, which implies that $tX^d + b_1X^{d-1} + \cdots + b_{d-1}X + b_d \in P$ since P is prime and $P \cap R = (0)$. Thus, since the degree of $f(X)$ is d, there exists $c \in R$ such that $cf(X) = tX^d + b_1X^{d-1} + \cdots + b_d$. Hence $t = ca_0 \in (a_0)$, which contradicts the assumption that $t \notin (a_0)$. $\square$

The converse statement of the above proposition holds (see [Sh]).

Corollary 2.1.4 *Let R be a factorial domain and P be a nonzero prime ideal in a polynomial ring $R[X]$ such that $P \cap R = (0)$. Then there exists an irreducible polynomial $f(X)$ with least positive degree among elements in P such that $P = f(X)R[X]$.*

Proof Choose $f(X) \in P$ as in Proposition 2.1.3 such that the greatest common divisor of the coefficients $a_0, \ldots, a_d$ of $f(X)$ is 1. Let there exist $t \notin a_0R$ such that $ta_i \in a_0R$ for all $i \geq 1$. Let $ta_i = b_ia_0$, $i = 1, \ldots, d$. This implies that $t \cdot \mathrm{GCD}(a_1, \ldots, a_d) = a_0 \cdot \mathrm{GCD}(b_1, \ldots, b_d)$. Hence a_0 divides t since $\mathrm{GCD}(a_0, a_1, \ldots, a_d) = 1$, which is a contradiction to the assumption that $t \notin a_0R$. $\square$

Corollary 2.1.5 *Let a, b be two nonzero elements of R. Then $(aX + b)R[X]$ is a prime ideal in $R[X]$ if and only if a, b is an R-sequence.*

Proof If $(aX + b)R[X] = P$ is a prime ideal, the proof is immediate by Proposition 2.1.3. Conversely, let a, b be an R-sequence. It is clear that $aX + b$ generates a prime ideal Q in $K[X]$. Let $P = Q \cap R[X]$. Then clearly P is

a prime ideal in $R[X]$ containing $aX + b$ such that $P \cap R = (0)$. The rest is immediate. $\square$

Corollary 2.1.6 *Let I be a nonzero ideal in $R[X]$ such that $I \cap R = (0)$. If there exists a polynomial $f(X) \in I$ such that $f(X)$ is of least positive degree in I and that $\mathsf{c}(f(X)) = R$, then I is generated by the polynomial $f(X)$.*

Proof Let $f(X) = a_0X^d + \cdots + a_d$. Let $t \in R$ be such that $ta_i \in (a_0)$ for all $1 \leq i \leq d$, then clearly $t \in (a_0)$ because the content $\mathsf{c}(f(X)) = R$. Now, the result follows as in Proposition 2.1.3. $\square$

Corollary 2.1.7 *Let R be a regular ring. Any prime ideal P of $R[X]$ such that $P \cap R = (0)$ is invertible in $R[X]$.*

Proof The result will be proved by showing that $P(R[X])_M$ is principal for every maximal ideal M of $R[X]$. Take any maximal ideal M in $R[X]$, then $M \cap R = Q$ is a prime ideal in R. We shall show that $PR_Q[X]$ is principal. If $Q = (0)$, then $R_Q = K$, the field of fractions of R. In this case, we can see the result trivially. Let $Q \neq (0)$. Then R_Q is a factorial domain. Therefore the result follows from Corollary 2.1.4. Thus P is invertible in $R[X]$. $\square$

Theorem 2.1.8 *The following statements are equivalent:*

(i) *$E^{(\alpha)}$ is a principal ideal of $R[X]$;*

(ii) *$I_{[\alpha]}$ is a principal ideal of R;*

(iii) *there exists a Sharma polynomial in $E^{(\alpha)}$ of degree d.*

If one of the above conditions holds, then $E^{(\alpha)}$ is generated by a Sharma polynomial.

Proof (iii) $\Rightarrow$ (i): Let $f(X)$ be a Sharma polynomial in $E^{(\alpha)}$ of degree d. Since $\deg \varphi_\alpha(X) = d$, this Sharma polynomial has the least degree. Thus $E^{(\alpha)}$ is principal by Proposition 2.1.3.
(i) $\Rightarrow$ (ii): Let $E^{(\alpha)} = f(X)R[X]$. Then $f(X)R[X] \supseteq I_{[\alpha]}\varphi_\alpha(X)R[X]$. Note that $E^{(\alpha)} \otimes_R K = f(X)K[X] = \varphi_\alpha(X)K[X]$ and hence that $\deg f(X) = \deg \varphi_\alpha(X) = d$. Take $a \in I_{[\alpha]}$. Then $a\varphi_\alpha(X) = bf(X)$. Let $f(X) = a_0X^d + \cdots + a_d$ with $a_i \in R$. Then $a = ba_0$, so that $I_{[\alpha]} \supseteq a_0R$ for some $b \in R$. Since $ba_0\eta_i = a\eta_i = ba_i$ $(1 \leq i \leq d)$, we have $a_0\eta_i = a_i \in R$. Hence we have $a_0 \in I_{[\alpha]}$, which implies that $I_{[\alpha]} = a_0R$.

(ii) $\Rightarrow$ (iii): Let $I_{[\alpha]} = bR$. Then $I_{[\alpha]}\varphi_\alpha(X)R[X] = b\varphi_\alpha(X)R[X] \subseteq E^{(\alpha)}$ and $b\eta_i \in R$ $(1 \le i \le d)$. Suppose that there exists $t \notin bR$ with $tb\eta_i \in bR$ $(1 \le i \le d)$. Then $t\eta_i \in R$ and hence $t \in I_{[\alpha]} = bR$, a contradiction. Thus we conclude that $b\varphi_\alpha(X) \in R[X]$ is a Sharma polynomial of degree d. $\square$

2.2 Anti-Integral Elements and Super-Primitive Elements

We extend the anti-integrality developed in Chapter 1 to the high degree case.

Definition 2.2.1 (i) $\alpha \in L$ is called an *anti-integral element* of degree d over R if $E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, we say that $R[\alpha]$ is an *anti-integral extension* of R.
(ii) $\alpha \in L$ is called a *super-primitive element* of degree d over R if $J_{[\alpha]} \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R)$. When α is a super-primitive element, we say that $R[\alpha]$ is a *super-primitive extension* of R.

Remark 2.2.2 (i) In Chapter 1, we studied the anti-integrality in the birational case, which is defined as follows: An element $\alpha \in K$ is called anti-integral over R if $R = R[\alpha] \cap R[\alpha^{-1}]$ $(:= R\langle\alpha\rangle)$. We knew that α is anti-integral over R in this sense if and only if $E^{(\alpha)}$ has a linear basis, that is,

$$E^{(\alpha)} = \sum (c_i X - d_i)R[X]$$

with $d_i/c_i = \alpha$ (Proof of Lemma 1.1.9). The last condition is equivalent to $E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X]$, where $\varphi_\alpha(X) = X - \alpha$. So $\alpha \in K$ is anti-integral over R in this sense if and only if α is an anti-integral element of degree 1 over R in the sense of Definition 2.2.1, that is, the anti-integrality defined in Definition 2.2.1 is equivalent to the one defined in Definition 1.1.4 in the case of degree 1 (i.e., birational).
(ii) It is immediate that $\alpha \in L$ is a super-primitive element of degree d over R if and only if α is a super-primitive element of degree d over R_p for any $p \in \mathrm{Spec}(R)$. Thus $R[\alpha]$ is a super-primitive extension of R if and only if $R[\alpha]_p$ is a super-primitive extension of R_p for all $p \in \mathrm{Spec}(R)$, where $R[\alpha]_p$ denotes the localization $S^{-1}R[\alpha]$ with $S = R \setminus p$, that is, $R[\alpha]_p = R[\alpha]_p$. $\square$

Proposition 2.2.3 *Let α be an element in an algebraic field extension L of K with $d = [K(\alpha) : K]$. Let $\pi : R[X] \longrightarrow R[\alpha]$ be the natural R-homomorphism. If $Ker(\pi)$ is generated by polynomials of degree d, then $Ker(\pi)$*

$= I\varphi_\alpha(X)R[X]$ for some divisorial ideal I of R. Moreover I coincides with $I_{[\alpha]}$.

Proof Assume that $f_i(X)$ $(1 \le i \le s)$ generate $\mathrm{Ker}(\pi)$. Then $f_i(X) = a_i\varphi_\alpha(X)$ with some $a_i \in R$. It is easy to see that $a_i\eta_j \in R$ for all i, j. Hence we have $a_i \in \bigcap_{i=1}^d I_{\eta_i} = I_{[\alpha]}$. Put $I = (a_1, \ldots, a_s)R$. Then $I \subseteq I_{[\alpha]}$. We see that $I\varphi_\alpha(X)R[X] \supseteq (f_1(X), \ldots, f_s(X))R[X] = \mathrm{Ker}(\pi) \supseteq I_{[\alpha]}\varphi_\alpha(X)R[X]$. Accordingly it follows that $I = I_{[\alpha]}$ and $\mathrm{Ker}(\pi) = I\varphi_\alpha(X)R[X]$. ∎

Lemma 2.2.4 *Let $f(X)$ be an element of a polynomial ring $R[X]$ and let $p \in Spec(R)$. Then $p \supseteq \mathsf{c}(f(X))$ if and only if $R_p[X]/f(X)R_p[X]$ is not flat over R_p.*

Proof The implication ($\Leftarrow$) follows from [M1,(20.F)].
($\Rightarrow$) Since $\mathsf{c}(f(X)) \subseteq p$, it follows that $pR[X]$ contains $f(X)$ and hence that $Q = pR[X]/f(X)R[X]$ is a prime ideal of $B := R[X]/f(X)R[X]$. Suppose that $B_p = R_p[X]/f(X)R_p[X]$ is flat over R_p. Then B_Q is obtained from B_p by localizing at QB_p. So $\mathrm{depth}(B_Q) \ge \mathrm{depth}(B_p)$, and so $\mathrm{depth}(B_Q) \ge \mathrm{depth}(R_p)$. It is easy to see that $\mathrm{depth}(B_{pB}) = \mathrm{depth}(B_Q)$ and $B_{pB} = R[X]_{pR[X]}/f(X)R[X]_{pR[X]}$. Since R is an integral domain, it follows that $\mathrm{depth}(B_{pB}) = \mathrm{depth}(R[X]_{pR[X]}) - 1 = \mathrm{depth}(R_p) - 1$, which is a contradiction. ∎

What will be considered in this chapter is based on the following result.

Theorem 2.2.5 *Assume that α is an anti-integral element of degree d over R. Then for $p \in Spec(R)$, the following are equivalent:*

(i) $rank_{k(p)}R[\alpha] \otimes_R k(p) \le d$;

(ii) $rank_{k(p)}R[\alpha] \otimes_R k(p) < \infty$;

(iii) $R[\alpha] \otimes_R k(p)$ *is not isomorphic to a polynomial ring* $k(p)[T]$;

(iv) $J_{[\alpha]} \not\subseteq p$;

(v) $pR[X] \not\supseteq E^{(\alpha)}$;

(vi) $R[\alpha]_p$ *is flat over* R_p.

Proof Since α is an anti-integral element over R, we have $E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X]$.
(iv) $\Rightarrow$ (vi): Since $R_p = (J_{[\alpha]})_p = (I_{[\alpha]})_p\mathsf{c}(\varphi_\alpha(X))_p$, the ideal $(I_{[\alpha]})_p$ is a principal ideal bR_p for some $b \in I_{[\alpha]}$. So $(E^{(\alpha)})_p = b\varphi_\alpha(X)R_p[X]$. We have $R[\alpha]_p \simeq R_p[X]/(E^{(\alpha)})_p = R_p[X]/b\varphi_\alpha(X)R_p[X]$. Thus $R[\alpha]_p$ is flat over R_p by Lemma 2.2.4 because $R_p = (J_{[\alpha]})_p = \mathsf{c}(b\varphi_\alpha(X))_p$.

(iv) $\Rightarrow$ (i): By the same argument as above, we have $R[\alpha]_p \simeq R_p[X]/(E^{(\alpha)})_p = R_p[X]/b\varphi_\alpha(X)R_p[X]$. Since $R_p = (J_{[\alpha]})_p = \mathsf{c}(b\varphi_\alpha(X))_p$, there exists i ($0 \le i \le d$) such that $b\eta_i \notin pR_p[X]$. We take i minimal among such ones. Then $b\varphi_\alpha(X) = bX^d + b\eta_1 X^{d-1} + \cdots + b\eta_d \equiv b\eta_i X^{d-i} + \cdots + b\eta_d \not\equiv 0 \pmod{pR_p[X]}$, which means that $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p) \le d - i \le d$.
(i) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iv): Note that $R[\alpha]_p/pR[\alpha]_p \simeq R_p[X]/(pR[X] + E^{(\alpha)})_p$. The ideal $(pR[X] + E^{(\alpha)})_p$ contains an element $f(X) \in R[X]$ such that $\mathsf{c}(f(X))_p = R_p$ since $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p) < \infty$. Indeed, if not, we conclude that $R[\alpha] \otimes_R k(p) \simeq k(p)[T]$, a polynomial ring, a contradiction. We may assume that $f(X) \in E^{(\alpha)}$. So $(E^{(\alpha)})_p = I_{[\alpha]}\varphi_\alpha(X)R_p[X]$ yields that $(J_{[\alpha]})_p = (I_{[\alpha]})_p \mathsf{c}(\varphi_\alpha(X))_p = R_p$.
(vi) $\Rightarrow$ (iv): Suppose that $J_{[\alpha]} \subseteq p$. Localizing at p, we may assume that R is a local ring (R, m). Consider the exact sequence:

$$0 \to E^{(\alpha)} \to R[X] \to R[\alpha] \to 0.$$

Then $E^{(\alpha)}$ is flat over R because $R[X]$ and $R[\alpha]$ are flat over R. The isomorphism $E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X] \simeq I_{[\alpha]}R[X]$ yields that $I_{[\alpha]}R[X]$ is flat over $R[X]$ and hence $I_{[\alpha]}$ is flat over R. Since R is local, $I_{[\alpha]} = bR$ for some $b \in I_{[\alpha]}$. So $J_{[\alpha]} = b\mathsf{c}(\varphi_\alpha(X))$ and $E^{(\alpha)} = b\varphi_\alpha(X)R[X]$. Thus $\mathsf{c}(b\varphi_\alpha(X)) \subseteq m$, and hence $R[\alpha]$ is not flat over R by Lemma 2.2.4.
(iv) $\Rightarrow$ (v): Since $J_{[\alpha]} = I_{[\alpha]}\mathsf{c}(\varphi_\alpha(X)) \not\subseteq p$, there exists $a \in I_{[\alpha]}$ such that $a\mathsf{c}(\varphi_\alpha(X)) = \mathsf{c}(a\varphi_\alpha(X)) \not\subseteq p$. Thus $a\varphi_\alpha(X) \notin pR[X]$ and therefore $E^{(\alpha)} \not\subseteq pR[X]$.
(v) $\Rightarrow$ (iv): Since $E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X]$, there exists $a \in I_{[\alpha]}$ such that $\mathsf{c}(a\varphi_\alpha(X)) \not\subseteq p$. So $J_{[\alpha]} = I_{[\alpha]}\mathsf{c}(\varphi_\alpha(X)) \not\subseteq p$.
(v) $\Rightarrow$ (iii): There exists $f(X) \in E^{(\alpha)}$ with $f(X) \notin pR[X]$. So $R[\alpha]/pR[\alpha] = (R/p)[\alpha']$, where α' denotes the residue class of α in $R[\alpha]/pR[\alpha]$, and $f(\alpha') = 0$. Thus α' is algebraic over R/p.
(iii) $\Rightarrow$ (v): Suppose that $E^{(\alpha)} \subseteq pR[X]$. Then $R[\alpha]/pR[\alpha] = (R[X]/E^{(\alpha)})/(pR[X]/E^{(\alpha)}) = R[X]/pR[X] = (R/p)[X]$, which is a polynomial ring over R/p. $\square$

Recall here Definition 1.2.1 for convenience:

Let A be an extension of R and let $p \in \mathrm{Spec}(R)$. We say that A is a *blowing-up* at p or p is a *blowing-up point* of A/R if the following two conditions are satisfied:

(i) $pA_p \cap R_p = pR_p$ (equivalently $pA \cap R = p$);
(ii) A_p/pA_p is isomorphic to a polynomial ring $(R_p/pR_p)[T]$.

Making use of the above definition, we get the following corollary to Theorem 2.2.5.

Corollary 2.2.6 *When α is an anti-integral element over R, the blowing-up locus $\{p \in Spec(R) \mid p$ is a blowing-up point of $R[\alpha]\}$ is given by $V(J_{[\alpha]})$, and is the same as $\{p \in Spec(R) \mid R[\alpha]_p$ is not flat over $R_p\}$.*

Proof This follows from Theorem 2.2.5 and Lemma 2.2.4. ∎

The next proposition gives rise to the relation between Sharma polynomials and the ideal $E^{(\alpha)}$.

Theorem 2.2.7 *(a) $R[\alpha]$ is not a blowing-up at any point in $Dp_1(R)$ if and only if $E^{(\alpha)}$ contains a Sharma polynomial.*
(b) $R[\alpha]$ is not a blowing-up at any point in $Spec(R)$ if and only if there exists a polynomial $f(X)$ in $E^{(\alpha)}$ such that $\mathsf{c}(f(X)) = R$.

Proof (*a*) Take $g_0(X) \in E^{(\alpha)} \setminus (0)$. If $g_0(X)$ is a Sharma polynomial, then we are done. Suppose that $g_0(X)$ is not a Sharma polynomial. Let $\{p_1, \ldots, p_t\}$ be the set of all members in $\mathrm{Dp}_1(R)$ satisfying $\mathsf{c}(g_0(X)) \subseteq p_i$. Such p_i exists by Theorem 2.1.2. Since $E^{(\alpha)} \nsubseteq pR[X]$ for any $p \in \mathrm{Dp}_1(R)$, there are $g_i(X) \in E^{(\alpha)}$ such that $\mathsf{c}(g_i(X)) \nsubseteq p_i$ $(1 \leq i \leq t)$. Put $N(0) := \deg(g_0(X))$ and $N(i) := N(i-1) + \deg(g_i(X)) + 1$ inductively. Let $f(X) := \sum g_i(X)X^{N(i)}$. Then $\mathsf{c}(f(X)) = \mathsf{c}(g_0(X)) + \cdots + \mathsf{c}(g_t(X))$. By the choice of p_i, there does not exist $p \in \mathrm{Dp}_1(R)$ such that $\mathsf{c}(f(X)) \subseteq p$. Whence $f(X)$ is a Sharma polynomial. Assume that $E^{(\alpha)}$ contains a Sharma polynomial. Then $E^{(\alpha)} \nsubseteq pR[X]$ for any $p \in \mathrm{Dp}_1(R)$ by Theorem 2.1.2. So a blowing-up does not occur for $R[\alpha]$ on $\mathrm{Dp}_1(R)$ over R.
(*b*) Let $E^{(\alpha)} = (f_1(X), \ldots, f_n(X))R[X]$. Take $p \in \mathrm{Spec}(R)$. Then $E^{(\alpha)} \nsubseteq pR[X]$. So there exists i such that $\mathsf{c}(f_i(X)) \nsubseteq p$. Put $N(0) = 0$ and $N(i) = N(i-1) + \deg(f_i(X)) + 1$, and let $f(X) = \sum f_i(X)X^{N(i)}$. Then $\mathsf{c}(f(X)) = \mathsf{c}(f_1(X)) + \cdots + \mathsf{c}(f_n(X)) = R$. The converse is obvious. ∎

By the following theorem, we see that a super-primitive element is an anti-integral element.

Theorem 2.2.8 *The following statements are equivalent:*

(i) *α is a super-primitive element of degree d;*

(ii) *α is an anti-integral element of degree d over R and $R_p[\alpha]$ is flat over R_p for all $p \in Dp_1(R)$;*

(iii) *α is an anti-integral element of degree d over R and $pR[X] \not\supseteq E^{(\alpha)}$ for all $p \in Dp_1(R)$;*

(iv) *α is an anti-integral element of degree d over R and there exists a Sharma polynomial in $E^{(\alpha)}$;*

(v) *${J_{[\alpha]}}^{-1} = R$, where ${J_{[\alpha]}}^{-1} := R :_K J_{[\alpha]}$.*

Proof (i) $\Rightarrow$ (ii): It is clear that $I_{[\alpha]}\varphi_\alpha(X)R[X] \subseteq E^{(\alpha)}$, and hence $I_{[\alpha]}R[X] \subseteq \varphi_\alpha(X)^{-1}E^{(\alpha)}$. Put $J = \varphi_\alpha(X)^{-1}E^{(\alpha)}$. Let $I_{[\alpha]}R[X] = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition of the ideal $I_{[\alpha]}R[X]$ and let $P_i = \sqrt{Q_i}$ $(1 \le i \le n)$. Assume that Q (resp. P) represents some Q_i (resp. P_i). Since $I_{[\alpha]}$ is a divisorial ideal of R, $I_{[\alpha]}R[X]$ is a divisorial ideal of $R[X]$, and hence $\mathrm{depth}(R[X]_P) = 1$. Put $p = P \cap R$. As $p \supseteq I_{[\alpha]}$, we see that $p \neq (0)$. Thus we have $P = pR[X]$ and $\mathrm{depth}(R_p) = 1$. Since α is a super-primitive element, $J_{[\alpha]} \not\subseteq p$ by definition. Therefore there exists an element $a \in I_{[\alpha]}$ such that $(E^{(\alpha)})_p = a\varphi_\alpha(X)R_p[X]$ by Theorem 2.1.8. Hence we have $J_p = aR_p[X] \subseteq I_{[\alpha]}R_p[X] \subseteq QR_p[X]$. Thus we get $J \subseteq R[X] \cap QR_p[X] = Q$, that is, $J \subseteq I_{[\alpha]}R[X]$ because Q (resp. P, p) is every Q_i (resp. P_i, $p_i := P_i \cap R$) for $1 \le i \le n$. This implies that α is an anti-integral element. Hence the assertion follows from Theorem 2.2.5.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv): These are immediate from Theorem 2.2.5 and Theorem 2.2.7.

(iv) $\Rightarrow$ (i): Since α is an anti-integral element, $E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X]$. By Theorem 2.2.7 (a), we have $E^{(\alpha)} \not\subseteq pR[X]$ for all $p \in \mathrm{Dp}_1(R)$. Hence there exists an element $a(p) \in I_{[\alpha]}$ such that $f(X) = a(p)\varphi_\alpha(X)$ and $\mathsf{c}(f(X)) \not\subseteq p$. Thus $J_{[\alpha]} \not\subseteq p$ for any $p \in \mathrm{Dp}_1(R)$. Therefore α is a super-primitive element over R.

(i) $\Leftrightarrow$ (v): Assume that $J_{[\alpha]} \not\subseteq p$ for any $p \in \mathrm{Dp}_1(R)$. Then $({J_{[\alpha]}}^{-1})_p = (R :_K J_{[\alpha]})_p = (R_p :_K (J_{[\alpha]})_p) = R_p :_K R_p = R_p$ for any $p \in \mathrm{Dp}_1(R)$. Since ${J_{[\alpha]}}^{-1}$ is a divisorial ideal of R, we have $R = \bigcap R_p = \bigcap ({J_{[\alpha]}}^{-1})_p \supseteq {J_{[\alpha]}}^{-1}$, where p ranges over the prime ideals of depth one. So it follows that $R = {J_{[\alpha]}}^{-1}$. Conversely, suppose that $R = {J_{[\alpha]}}^{-1}$ and $J_{[\alpha]} \subseteq p$ for some $p \in \mathrm{Dp}_1(R)$. Then ${J_{[\alpha]}}^{-1} \supseteq p^{-1}$ and hence $R = ({J_{[\alpha]}}^{-1})^{-1} \subseteq (p^{-1})^{-1} = p$, a contradiction. $\square$

By the following result, we see that a super-primitive element is not so special.

Theorem 2.2.9 *Assume that R is a Krull domain. Then any element α which is algebraic over R is a super-primitive element over R.*

Proof Since R is a Krull domain, $\mathrm{Dp}_1(R) = \mathrm{Ht}_1(R)$. Take $p \in \mathrm{Ht}_1(R)$. Then R_p is a DVR. Let v denote the valuation corresponding to R_p. Let $\varphi_\alpha(X) =$

$X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ be the minimal polynomial of α. Put $\eta_0 = 1$. Then there exists j such that $v(\eta_j) \leq v(\eta_i)$ for all i. Thus $\eta_i/\eta_j = a_i/b \in R_p$, where $b \in R \setminus p$, $a_i \in R$. In particular, $a_j = b \notin p$. Hence

$$\varphi_\alpha(X) = \eta_j(a_0/\eta_j)X^d + \cdots + \eta_j(a_d/\eta_j)\eta_d$$

Note that $I_{[\alpha]}R_p = I_{\eta_j}R_p$ by the choice of η_j. Hence $f(X) := (b/\eta_j)\varphi_\alpha(X) = a_0X^d + \cdots + a_d \in I_{\eta_j}\varphi_\alpha(X)R_p[X] = I_{[\alpha]}\varphi_\alpha(X)R_p[X]$. Since $a_j = b \notin p$, we have $\mathsf{c}(f(X)) \not\subseteq pR_p$. Thus $J_{[\alpha]} \not\subseteq p$, which means that α is a super-primitive element over R. ∎

Once we find a super-primitive element, we can get many such elements. Indeed we obtain the following.

Proposition 2.2.10 *Assume that α is a super-primitive element of degree d over R. Then for any unit u of R and any element $b \in R$, $\beta = u\alpha + b$ is a super-primitive element of degree d over R.*

Proof We may assume that $u = 1$. It is clear that $\varphi_\beta(X) = \varphi_\alpha(X - b)$ because $K(\beta) = K(\alpha)$, $d = \deg \varphi_\alpha(X - b)$, and $\varphi_\alpha(X - b)$ is monic in $K[X]$. We see that $I_{[\alpha]} \subseteq I_{[\beta]}$ and $\mathsf{c}(\varphi_\alpha(X)) = \mathsf{c}(\varphi_\alpha(X - b)) = \mathsf{c}(\varphi_\beta(X))$. Since $(J_{[\alpha]})_p = (I_{[\alpha]})_p\mathsf{c}(\varphi_\alpha(X))_p = R_p$ for any $p \in \mathrm{Dp}_1(R)$ by Theorem 2.2.8, we have $R_p = (J_{[\alpha]})_p \subseteq (J_{[\beta]})_p$ and hence $(J_{[\beta]})_p = R_p$ for any $p \in \mathrm{Dp}_1(R)$. Thus we conclude that β is a super-primitive element of degree d over R by Theorem 2.2.8. ∎

Proposition 2.2.11 *Assume that R is a local ring containing an infinite field k and that $J_{[\alpha]} = R$. Then there exists an element $\lambda \in k$ which satisfies the following three conditions:*

(a) $1/(\alpha - \lambda)$ *belongs to* $R[\alpha]$*;*

(b) $1/(\alpha - \lambda)$ *is a super-primitive element of degree d over R;*

(c) $1/(\alpha - \lambda)$ *is integral over R.*

Proof Since R is local, there exists an element λ in k such that $I_{[\alpha]}\varphi_\alpha(X + \lambda)$ contains a degree d polynomial $g(X)$ in $R[X]$ of which constant term is 1. Put $\beta = \alpha - \lambda$. Then $g(\beta) = 0$. Put $h(X) = X^d g(1/X) \in R[X]$. Then $h(\beta^{-1}) = (\beta^{-1})^d g(\beta) = 0$. So β^{-1} is integral over R. Since $[K(\alpha) : K] = [K(\beta) : K] = d$, we conclude that $\varphi_{\beta^{-1}}(X) = h(X) \in R[X]$. Thus $I_{[\beta^{-1}]} = R$ and hence $J_{[\beta^{-1}]} = I_{[\beta^{-1}]}\mathsf{c}(\varphi_{\beta^{-1}}(X)) = R$. In particular, β^{-1} is a super-primitive element of degree d over R by Theorem 2.2.8. ∎

2.3 Integrality and Flatness of Anti-Integral Extensions

For $p \in \mathrm{Spec}(R)$, $k(p)$ denotes the residue field R_p/pR_p and $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p)$ denotes the dimension as a vector space over $k(p)$. We are interested in characterizing the flatness and the integrality of an anti-integral extension $R[\alpha]$ of R. Among others, we obtain the following:

$R[\alpha]$ is integral over R if and only if $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p) = d$ for all $p \in \mathrm{Spec}(R)$. Thus if an anti-integral extension $R[\alpha]$ is integral over R, then $R[\alpha]$ is flat over R (Theorem 2.2.5).

The following result asserts that the integrality of an extension of R is determined by localizing at the prime ideals in $\mathrm{Dp}_1(R)$.

Proposition 2.3.1 *Let A be an integral domain containing R. Then A is integral over R if and only if A_p $(:= A \otimes_R R_p)$ is integral over R_p for any $p \in Dp_1(R)$.*

Proof The implication ($\Rightarrow$) is trivial. Now consider the converse. We assume that A_p is integral over R_p for any $p \in \mathrm{Dp}_1(R)$. It suffices to show that α is integral over R. Let $\overline{R}$ be the integral closure of R in K. Then $\overline{R}$ is a Krull domain [M1, p.144]. It suffices to show that α is integral over $\overline{R}$. Let $\tilde{R}$ be the integral closure of R in $K(A)$ and let $C = \tilde{R} :_{\tilde{R}} \alpha$, a denominator ideal of $\tilde{R}$. Then $K(\tilde{R}) = K(A)$ and C is a divisorial ideal of $\tilde{R}$. There exists $P \in \mathrm{Dp}_1(\tilde{R}) = \mathrm{Ht}_1(\tilde{R})$ such that $C \subseteq P$. Since $\tilde{R}/\overline{R}$ is integral and $\overline{R}$ is integrally closed in K, the Going-Down Theorem, holds between $\tilde{R}$ and $\overline{R}$. Thus $P \cap \overline{R} \in \mathrm{Ht}_1(\overline{R}) = \mathrm{Dp}_1(\overline{R})$. In particular, $P \cap \overline{R}$ is a divisorial ideal of $\overline{R}$. So $\tilde{R} :_{\overline{R}} \alpha = C \cap \overline{R} \subseteq P \cap \overline{R} \in \mathrm{Dp}_1(\overline{R})$. By [F,(4.6)], $(P \cap \overline{R}) \cap R$ is a divisorial ideal of R. Hence $\tilde{R} :_R \alpha = (C \cap \overline{R}) \cap R \subseteq (P \cap \overline{R}) \cap R \in \mathrm{Dp}_1(R)$. Put $p = (P \cap \overline{R}) \cap R$. Then we have $p \in \mathrm{Dp}_1(R)$ and $\tilde{R} :_R \alpha \subseteq p$, which is a contradiction. $\square$

The integrality of anti-integral extensions is characterized as follows:

Theorem 2.3.2 *Assume that α is an anti-integral element of degree d over R. Then the following statements are equivalent:*

(i) *$R[\alpha]$ is integral over R;*

(ii) *$\varphi_\alpha(X) \in R[X]$;*

(iii) *$I_{[\alpha]} = R$;*

(iv) *$rank_{k(q)} R[\alpha] \otimes_R k(q) = d$ for any $q \in Dp_1(R)$;*

(v) *$rank_{k(q)} R[\alpha] \otimes_R k(q) = d$ for any $q \in Spec(R)$.*

Proof Since α is anti-integral, $E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X]$. So the equivalence of (i), (ii), and (iii) are immediate because $R[X]/E^{(\alpha)} \simeq R[\alpha]$, and implications (ii) $\Rightarrow$ (v) $\Rightarrow$ (iv) are obvious.
(iv) $\Rightarrow$ (ii): Suppose that $I_{[\alpha]} \subseteq p$ for some $p \in \mathrm{Dp}_1(R)$. Since $J_{[\alpha]} = I_{[\alpha]}\mathsf{c}(\varphi_\alpha(X)) \not\subseteq p$ by Theorem 2.2.5, $(I_{[\alpha]})_p$ is an invertible ideal of R_p and hence $(I_{[\alpha]})_p$ is a principal ideal bR_p of R_p for some b. Whence we have $(E^{(\alpha)})_p = (I_{[\alpha]})_p\varphi_\alpha(X)R_p[X] = (b\varphi_\alpha(X))R_p[X]$. Since $I_{[\alpha]} \subseteq p$, $b\varphi_\alpha(X) \in R_p[X]$ is not monic. Hence either $R[\alpha] \otimes_R k(p) \simeq k(p)[T]$, a polynomial ring or $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p) < d$ holds, a contradiction. ∎

By the above theorem, we see that the obstruction of integrality of anti-integral extensions is given by $I_{[\alpha]}$. Namely, we obtain the following.

Corollary 2.3.3 *Assume that α is an anti-integral element over R. Then $V(I_{[\alpha]}) = \{p \in \mathrm{Spec}(R) | R[\alpha]_p$ is not integral over $R_p\}$.*

Proof It is easy to see that the integrality is a local-global property. So our conclusion follows from Theorem 2.3.2. ∎

Remark 2.3.4 Let R be a Noetherian normal domain and let α be an element in a field L containing R. If α is integral over R, then $R[\alpha]$ is flat and super-primitive over R. Indeed, when $\varphi_\alpha(X) \in K[X]$ denotes the minimal polynomial of α over R, it is known that α is integral over R if and only if $\varphi_\alpha(X)$ belongs to $R[X]$ ([M2, (9.2)]). Since R is normal, $p \in \mathrm{Dp}_1(R) \Rightarrow \mathrm{ht}(p) = 1 \Rightarrow R_p$ is a DVR. As $R[\alpha]$ is a finite R-module, $R[\alpha]_p$ is free over R_p for any $p \in \mathrm{Dp}_1(R)$. By Theorem 2.2.9, α is a super-primitive element over R. Moreover $R[\alpha]$ is flat over R by Theorems 2.2.5 and 2.3.2 because $R[\alpha]$ is a super-primitive, integral, and flat extension of R. ∎

Summing up the results in the preceding argument, we obtain the following:
Assume that α is an anti-integral element of degree d. Let p be a prime ideal of R. Then

(1) $R[\alpha]_p$ is flat over R_p if and only if $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p) \leq d$;

(2) $R[\alpha]_p$ is integral over R_p if and only if $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p) = d$.

In particular, we conclude:

Corollary 2.3.5 *Assume that α is an anti-integral element of degree d. If $R[\alpha]$ is integral over R, then $R[\alpha]$ is flat over R.*

In view of Theorem 2.2.7, we extend Theorem 2.2.5 to the following.

Theorem 2.3.6 *Assume that α is an anti-integral element of degree d over R. Then the following are equivalent:*

(i) *$R[\alpha]$ is flat over R;*
(ii) *$J_{[\alpha]} = R$;*
(iii) *$rank_{k(p)} R[\alpha] \otimes_R k(p) < \infty$ for any $p \in Spec(R)$;*
(iv) *$rank_{k(p)} R[\alpha] \otimes_R k(p) \leq d$ for any $p \in Spec(R)$;*
(v) *$R[\alpha]$ is not a blowing-up at any point in $Spec(R)$;*
(vi) *$R[\alpha]$ is quasi-finite over R;*
(vii) *$E^{(\alpha)}$ contains a polynomial $f(X)$ with $\mathsf{c}(f(X)) = R$.*

Proof The proof follows from Theorems 2.2.5 and 2.2.7(b). ∎

Remark 2.3.7 Let A be an over-ring of R (i.e., $R \subseteq A$ and $K(A) = K$). If A is integral and flat over R on $\mathrm{Dp}_1(R)$, then $A = R$. Indeed, it is known that $R = \bigcap_{p \in \mathrm{Dp}_1(R)} R_p$. For $p \in \mathrm{Dp}_1(R)$, A_p is integral and flat over R_p by the assumption. So A_p is a free R_p-module of rank 1. Thus $A_p = R_p$ and hence $R = \bigcap_{p \in \mathrm{Dp}_1(R)} R_p \supseteq A$, which means that $R = A$. ∎

Relating to this remark, we have the following.

Theorem 2.3.8 *Let α be an algebraic element over R. If $R[\alpha]$ is integral and flat at any point in $Dp_1(R)$, then $R[\alpha]$ is a free R-module and α is a super-primitive element over R. In particular, if α is both anti-integral and integral over R, then $R[\alpha]$ is a free R-module $R \oplus R\alpha \oplus \cdots \oplus R\alpha^{d-1}$.*

Proof First, we shall show that $I_{[\alpha]} = R$. Suppose that $I_{[\alpha]} \neq R$. Then there exists $p \in \mathrm{Dp}_1(R)$ such that $I_{[\alpha]} \subseteq p$ because $I_{[\alpha]}$ is a divisorial ideal of R. Since $R[\alpha]_p$ is integral over R_p by assumption, $R[\alpha]_p$ is a flat extension of R_p. As $R[\alpha]_p$ is flat over R_p, $R[\alpha]_p$ is a free R_p-module of rank d. Here we want to show that $R[\alpha]_p = R_p + R_p\alpha + \cdots + R_p\alpha^{d-1}$. To this end, we have only to show that $1', \alpha', \ldots, \alpha'^{d-1} \in R[\alpha]_p/pR[\alpha]_p$ are linearly independent over $k(p)$, where α' denotes the residue class of α in $R[\alpha]_p/pR[\alpha]_p$. [Suppose the

contrary. Then $R[\alpha]_p/pR[\alpha]_p = k(p)[\alpha'] = k(p) + k(p)\alpha' + \cdots + k(p)\alpha'^s$ for some $s < d$. But $R[\alpha]_p$ is a free R_p-module of rank d, which asserts that $\mathrm{rank}_{k(p)} R[\alpha]_p/pR[\alpha]_p = d$, a contradiction.] Thus we have shown that $R[\alpha]_p = R_p + R_p\alpha + \cdots + R_p\alpha^{d-1}$. So we have a relation:

$$\alpha^d = \lambda_0 + \lambda_1\alpha + \cdots + \lambda_{d-1}\alpha^{d-1}$$

($\lambda_i \in R_p$). Since the minimal monic polynomial $\varphi_\alpha(X)$ of α is unique, we have $\varphi_\alpha(X) = X^d - \lambda_{d-1}X^{d-1} - \cdots - \lambda_0$. So $I_{[\alpha]} \not\subseteq p$, a contradiction. Thus $\varphi_\alpha(X) \in R[X]$, which implies that $E^{(\alpha)} = \varphi_\alpha(X)R[X]$ and $R[\alpha]$ is free R-module. Since $\mathrm{c}(\varphi_\alpha(X)) = R$, we conclude that $J_{[\alpha]} = R$, which asserts that α is a super-primitive element over R by Theorem 2.2.8. ∎

Now we consider a certain over-ring of R, which is seen in Chapter 1. Let J be a fractional ideal of R. Recall $\mathcal{R}(J) := J :_K J$, which is an over-ring of R.

Lemma 2.3.9 *Let J be a divisorial ideal of R. Then $\mathcal{R}(J) = R$ if and only if $\mathcal{R}(J^{-1}) = R$.*

Proof Since J is divisorial, we have $(J^{-1})^{-1} = J$. So it suffices to prove one of the implications. Assume that $\mathcal{R}(J) = R$. The implication $\mathcal{R}(J^{-1}) \supseteq R$ is obvious. Take $\lambda \in \mathcal{R}(J^{-1})$. Then $\lambda J^{-1} \subseteq J^{-1}$. Thus $R :_K \lambda J^{-1} \supseteq R :_K J^{-1} = (J^{-1})^{-1} = J$. On the other hand, we have $R : \lambda J^{-1} = \lambda^{-1}R :_K J^{-1} = \lambda^{-1}(R :_K J^{-1}) = \lambda^{-1}(J^{-1})^{-1} = \lambda^{-1}J$. Thus $\lambda^{-1}J \supseteq J$, which shows that $J \supseteq \lambda J$, and hence $\lambda \in \mathcal{R}(J) = R$. ∎

By these arguments, we extend Theorem 2.2.8 as follows:

Theorem 2.3.10 *The following conditions are equivalent:*

(i) *α is a super-primitive element over R;*

(ii) *for each $p \in Dp_1(R)$, there exists $f(X) \in E^{(\alpha)}$ with $(E^{(\alpha)})_p = f(X)R_p[X]$;*

(iii) *for each $p \in Dp_1(R)$, there exists $a \in I_{[\alpha]}$ with $(I_{[\alpha]})_p = aR_p$;*

(iv) *$\mathcal{R}(I_{[\alpha]}) = R$.*

Proof Denote the degree of α by d.
(i) $\Rightarrow$ (ii) : Since $J_{[\alpha]} = I_{[\alpha]}\mathrm{c}(\varphi_\alpha(X)) \not\subseteq p$ for any $p \in \mathrm{Dp}_1(R)$, there exists $a \in I_{[\alpha]}$ with $f(X) := a\varphi_\alpha(X) \in pR[X]$. Note that $(E^{(\alpha)})_K \cap R_p[X] = (E^{(\alpha)})_p$ and $f(X) \in (E^{(\alpha)})_p$. By Theorem 2.1.2, $f(X)$ is a Sharma polynomial of degree d in $R_p[X]$. So $(E^{(\alpha)})_p = f(X)R_p[X]$.
(ii) $\Rightarrow$ (iii): Suppose that $(E^{(\alpha)})_p = f(X)R_p[X]$. Then $\deg f(X) = d$. Let a be the leading coefficient of $f(X)$. Then $\varphi_\alpha(X) = (1/a)f(X)$ by the uniqueness

of the minimal polynomial of α. So $f(X) = a\varphi_\alpha(X)R[X]$, and hence $a \in I_{[\alpha]}$. Since $(E^{(\alpha)})_p = f(X)R_p[X]$, $(I_{[\alpha]})_p = aR_p$.
(iii) $\Leftrightarrow$ (iv): We know that $\mathcal{R}(I_{[\alpha]}) = R$ if and only if $\mathcal{R}({I_{[\alpha]}}^{-1}) = R$ by Lemma 2.3.9. So apply the result of Proposition 1.3.2 and we conclude that (iii) and (iv) are equivalent.
(iii) $\Rightarrow$ (i): Since $(I_{[\alpha]})_p$ is a principal ideal of R_p for any $p \in \mathrm{Dp}_1(R)$, there exists $f(X) \in E^{(\alpha)}$ such that $\deg f(X) = d$ and $(E^{(\alpha)})_p = f(X)R_p[X]$. Since $f(X)$ is a Sharma polynomial in $R_p[X]$ by Theorem 2.1.2 and $\mathrm{depth}(R_p) = 1$, $\mathsf{c}(f(X)) \not\subseteq p$. Thus $J_{[\alpha]} \not\subseteq p$ for any $p \in \mathrm{Dp}_1(R)$ and hence α is a super-primitive element over R by definition. $\square$

2.4 Anti-Integrality of α and α^{-1}

One objective of this section is to show that the anti-integrality is also characterized by the set $\mathrm{Dp}_1(R)$. In fact, we prove the following statements are equivalent:

(1) α is an anti-integral element over R;

(2) α is an anti-integral element over R_p for all $p \in \mathrm{Dp}_1(R)$.

The other objective is to study several conditions equivalent to the statement that α is an anti-integral element of degree d over R by use of the ring $R[\alpha] \cap R[\alpha^{-1}]$. Put $R\langle\alpha\rangle := R[\alpha] \cap R[\alpha^{-1}]$. We begin with the following theorem, which characterizes anti-integrality.

Theorem 2.4.1 *The following statements are equivalent:*

(1) *α is an anti-integral element of degree d over R;*

(2) *the ideal $I_{[\alpha]}\eta_d$ of R is generated by the set $\{g(0) \mid g(X) \in E^{(\alpha)}\}$.*

Proof (1) $\Rightarrow$ (2): Let J be the ideal of R generated by the set $\{g(0) \mid g(X) \in E^{(\alpha)}\}$. Since $I_{[\alpha]}\varphi_\alpha(X) \subseteq E^{(\alpha)}$ and the constant term of $I_{[\alpha]}\varphi_\alpha(X)$ is $I_{[\alpha]}\eta_d$, it follows that $I_{[\alpha]}\eta_d \subseteq J$. Conversely take $a \in J$, and let

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a = 0$$

be a relation, where $a_i \in R$ and $a_n \neq 0$. Put $f(X) = a_nX^n + \cdots + a_1X + a$. Then $f(X) \in E^{(\alpha)}$. Since α is an anti-integral element of degree d over R, we have $E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Hence $f(X) = \sum h_i(X)g_i(X)$ for some $h_i(X) \in$

$I_{[\alpha]}\varphi_\alpha(X)$ and $g_i(X) \in R[X]$. Thus $a = f(0) = \sum h_i(0)g_i(0) \in I_{[\alpha]}\eta_d$, as desired.
(2) $\Rightarrow$ (1): Let $0 \neq f(X) \in E^{(\alpha)}$ and write $f(X) = a_nX^n + \cdots + a_1X + a$. Since $[K(\alpha) : K] = d$, we have $n \geq d$. By the assumption that $a \in J = I_{[\alpha]}\eta_d$, it follows that $a = b\eta_d$ for some $b \in I_{[\alpha]}$. Put $g(X) = bX^d + (b\eta_1)X^{d-1} + \cdots + (b\eta_d)$. Note that $g(X) \in E^{(\alpha)}$. As $f(0) = g(0)$, we get $f(X) - g(X) = X(h(X)) \in E^{(\alpha)}$ for some $h(X) \in R[X]$. Since $R[\alpha]$ is an integral domain, $E^{(\alpha)}$ is a prime ideal of $R[X]$, and hence $h(X) \in E^{(\alpha)}$. Since $\deg h(X) \leq n - 1$, we can prove $f(X) \in I_{[\alpha]}\varphi_\alpha R[X]$ by induction. Therefore α is an anti-integral element of degree d over R. $\square$

Remark 2.4.2 A similar result to Theorem 2.4.1 is valid when R is a Noetherian reduced ring and α is algebraic over R and is not a zero-divisor in $R[\alpha]$. $\square$

Under this preparation, we obtain the following result mentioned before.

Theorem 2.4.3 *The following statements are equivalent to each other:*

(1) *α is an anti-integral element of degree d over R;*

(2) *α is an anti-integral element of degree d over R_p for all $p \in Dp_1(R)$.*

Proof (1) $\Rightarrow$ (2): By the assumption, we have the following exact sequence:

$$0 \to I_{[\alpha]}\varphi_\alpha(X)R[X] \to R[X] \to R[\alpha] \to 0$$

Take $p \in \mathrm{Dp}_1(R)$. Tensoring $- \otimes_R R_p$, we have an exact sequence:

$$0 \to I_{[\alpha]}\varphi_\alpha(X)R_p[X] \to R_p[X] \to R_p[\alpha] \to 0$$

This exact sequence implies that α is an anti-integral element of degree d over R_p.

(2) $\Rightarrow$ (1): Consider the following canonical exact sequence:

$$0 \to E^{(\alpha)} \to R[X] \to R[\alpha] \to 0$$

Let J denote the ideal generated by the set $\{g(0) \mid g(X) \in E^{(\alpha)}\}$. We need to show that $I_{[\alpha]}\eta_d = J$ by (2). Since $I_{[\alpha]}\varphi_\alpha(X) \subseteq E^{(\alpha)}$, we have $I_{[\alpha]}\eta_d \subseteq J$. We shall show the converse inclusion. Since α is an anti-integral element of degree d over R_p by assumption, we conclude that $(I_{[\alpha]}\eta_d)_p = J_p$ for all $p \in \mathrm{Dp}_1(R)$ by (2). Thus we have $J \subseteq J_p = (I_{[\alpha]}\eta_d)_p$. Let $q \in \mathrm{Spec}(R)$ be a prime divisor of $I_{[\alpha]}\eta_d$. Since $I_{[\alpha]}\eta_d$ is a divisorial ideal, we see that $q \in \mathrm{Dp}_1(R)$ ([Y, Proposition 5.6]). Hence $J \subseteq \bigcap_{p\in \mathrm{Dp}_1(R)}(I_{[\alpha]}\eta_d)_p = I_{[\alpha]}\eta_d$ ([Y, Proposition 5.6]), which completes the proof. $\square$

The rest of this section, we investigate several conditions for α to be an anti-integral element over R by use of the properties of the ring $R\langle\alpha\rangle = R[\alpha] \cap R[\alpha^{-1}]$.

Lemma 2.4.4 *The following assertions are valid:*

(1) *if* $\{a \in R \mid a\alpha \in R\langle\alpha\rangle\} = I_{[\alpha]}$, *then* α *is an anti-integral element of degree* d *over* R;

(2) *if* α^{-1} *is an anti-integral element of degree* d *over* R, *then* $\{a \in R \mid a\alpha \in R\langle\alpha\rangle\} = I_{[\alpha]}$.

Proof (1) Take $f(X) \in R[X]$ satisfying $f(\alpha) = 0$, i.e., $f(X) \in E^{(\alpha)}$. We must show that $f(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$ by induction on $\deg f(X)$. Put $\deg f(X) = n$. If $n = d$, then $f(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$. Suppose that $n > d$. Put $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0$. Then $a_n\alpha + a_{n-1} \in R[\alpha] \cap R[\alpha^{-1}]$. By induction hypothesis, we have $a_n \in I_{[\alpha]}$. Put $g(X) = f(X) - a_nX^{n-d}\varphi_\alpha(X) \in R[X]$. We see that $g(\alpha) = 0$ and $\deg g(X) \leq n-1$. By induction hypothesis, we have that $g(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$. Hence $f(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$. Since $I_{[\alpha]}R[X] \subseteq E^{(\alpha)}$ is obvious, we have $E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Therefore α is an anti-integral element of degree d over R.
(2) To prove the opposite inclusion, take $a \in S$, then $a\alpha \in R\langle\alpha\rangle$. Thus $-a\alpha = c_0 + c_1\alpha^{-1} + \cdots + c_n(\alpha^{-1})^n$ where $c_0, c_1, \ldots, c_n$ are elements of R. Hence $c_n(\alpha^{-1})^{n+1} + \cdots + c_0\alpha^{-1} + a = 0$. Since α^{-1} is an anti-integral element of degree d over R, there exist $F_i(X) \in R[X]$ and $G_i(X) \in R[X]$ such that $f(X) = \sum F_i(X)G_i(X)$ where $\deg F_i(X) = d$ and $F_i(\alpha^{-1}) = 0$. Therefore $a = f(0) = \sum F_i(0)G_i(0)$. Let $F_i(X) = a_dX^d + \cdots + a_1X + a_0$. It follows that $a_d(\alpha^{-1})^d + \cdots + a_1\alpha^{-1} + a_0 = 0$. Hence $a_0\alpha^d + a_1\alpha^{d-1} + \cdots + a_d = 0$. Since $\eta_i = a_i/a_0$, we see that $a_0 = F_i(0) \in \bigcap_{i=1}^d I_{\eta_i} = I_{[\alpha]}$. Consequently, we get that $a = \sum F_i(0)G_i(0) \in I_{[\alpha]}$. Take any element a of $I_{[\alpha]}$. Put $a_i = a\eta_i$. Then we have that $a\alpha^d + a_1\alpha^{d-1} + \cdots + a_d = 0$ ($a_i \in R$). Hence $a\alpha = -(a_1 + a_2\alpha^{-1} + \cdots + a_d(\alpha^{-1})^{d-1}) \in R\langle\alpha\rangle$. Putting $S = \{a \in R \mid a\alpha \in R\langle\alpha\rangle\}$, we see that $I_{[\alpha]} \subseteq S$. ∎

By use of the above lemma, we have the following result.

Theorem 2.4.5 *An element* α *is an anti-integral element of degree* d *over* R *if and only if so is* α^{-1}.

Proof Suppose that α^{-1} is anti-integral. By Lemma 2.4.4(2), it follows that $\{a \in R \mid a\alpha \in R[\alpha] \cap R[\alpha^{-1}]\} = I_{[\alpha]}$. By Lemma 2.4.4(1), α is anti-integral. By this result, the converse is also valid. ∎

Another Proof of Theorem 2.4.5 Let $\pi_\alpha : R[X] \to R[\alpha]$, $\pi_{\alpha^{-1}} : R[X^{-1}] \to R[\alpha^{-1}]$ and $\pi : R[X, X^{-1}] \to R[\alpha, \alpha^{-1}]$ be R-algebra homomorphism sending X to α. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ker}(\pi) \cap R[X^{-1}] & \to & R[X^{-1}] & \to & R[\alpha^{-1}] & \to & 0 \\
 & & \cap & & \cap & & \cap & & \\
0 & \to & \mathrm{Ker}(\pi) & \to & R[X, X^{-1}] & \to & R[\alpha, \alpha^{-1}] & \to & 0 \\
 & & \cup & & \cup & & \cup & & \\
0 & \to & \mathrm{Ker}(\pi) \cap R[X] & \to & R[X] & \to & R[\alpha] & \to & 0
\end{array}$$

It is easy to see that $\mathrm{Ker}(\pi_{\alpha^{-1}}) = \mathrm{Ker}(\pi) \cap R[X^{-1}]$ and $\mathrm{Ker}(\pi_\alpha) = \mathrm{Ker}(\pi) \cap R[X]$. It is also easy to see that $\mathrm{Ker}(\pi) = (\mathrm{Ker}(\pi) \cap R[X])R[X, X^{-1}] = (\mathrm{Ker}(\pi) \cap R[X^{-1}])R[X, X^{-1}]$. Now we claim:

$$I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(X^{-1})R[X, X^{-1}] = I_{[\alpha]}\varphi_\alpha(X)R[X, X^{-1}]$$

[Indeed, we see that $I_{\alpha^{-1}}\varphi_{\alpha^{-1}}(X^{-1}) = \eta_d I_{[\alpha]}\eta_d^{-1}\varphi_\alpha(X)X^{-d}$. Hence it follows that $I_{\alpha^{-1}}\varphi_{\alpha^{-1}}(X^{-1})R[X, X^{-1}] = I_{[\alpha]}\varphi_\alpha(X)X^{-d}R[X, X^{-1}] = I_{[\alpha]}\varphi_\alpha(X)R[X, X^{-1}]$.] Since α is anti-integral over R, we have

$$\mathrm{Ker}(\pi_\alpha) = I_{[\alpha]}\varphi_\alpha(X)R[X]$$

Thus

$$\begin{aligned}
& \mathrm{Ker}(\pi_{\alpha^{-1}})R[X, X^{-1}] \\
= \; & (\mathrm{Ker}(\pi) \cap R[X^{-1}])R[X, X^{-1}] \\
= \; & I_{[\alpha]}\varphi_\alpha(X)R[X, X^{-1}] \\
= \; & I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(X^{-1})R[X, X^{-1}]
\end{aligned}$$

Hence

$$\begin{aligned}
& \mathrm{Ker}(\pi_{\alpha^{-1}}) \\
= \; & \mathrm{Ker}(\pi_{\alpha^{-1}})R[X, X^{-1}] \cap R[X^{-1}] \\
= \; & I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(X^{-1})R[X, X^{-1}] \cap R[X^{-1}] \\
= \; & I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(X^{-1})R[X^{-1}]
\end{aligned}$$

Therefore we conclude that $\mathrm{Ker}(\pi_{\alpha^{-1}}) = I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(X^{-1})R[X^{-1}]$, which means that α^{-1} is anti-integral over R. $\square$

Now we shall see that an analogous result of Theorem 2.4.5 holds for superprimitiveness.

Note that $I_{[\alpha]} = \bigcap_{i=1}^d I_{\eta_i}$. Since $\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0$, we have $1 + \eta_1(\alpha^{-1}) + \cdots + \eta_d(\alpha^{-1})^d = 0$, and hence $(\alpha^{-1})^d + \eta_d^{-1}\eta_{d-1}(\alpha^{-1})^{d-1} +$

$\cdots + \eta_d^{-1} = 0$. So $\varphi_{\alpha^{-1}}(X) = X^d + \eta_d^{-1}\eta_{d-1}C^{d-1} + \cdots + \eta_d^{-1}\eta_1 X + \eta_d^{-1}$. Hence $I_{[\alpha^{-1}]} = (\bigcap_{i=1}^{d-1} I_{\eta_d^{-1}\eta_i}) \cap I_{\eta_d^{-1}}$.

Lemma 2.4.6 *For an element $\alpha \in L$, $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$.*

Proof Take $\eta_d x \in \eta_d I_{[\alpha]}$ with $x \in I_{[\alpha]}$. Then $\eta_d^{-1} \cdot \eta_d x = x \in R$ and $\eta_d^{-1}\eta_i \cdot \eta_d x = x\eta_i \in R$. Thus $\eta_d x \in I_{[\alpha^{-1}]}$. Conversely, take $y \in I_{[\alpha^{-1}]} = (\bigcap_{i=1}^{d-1} I_{\eta_d^{-1}\eta_i}) \cap I_{\eta_d^{-1}}$. Then $\eta_d^{-1} y \in R$. Put $x := \eta_d^{-1} y$. Then $x\eta_d = y \in R$. We have that $x\eta_i = \eta_d^{-1}\eta_i\eta_d x = (\eta_d^{-1}\eta_i)y \in R$. So we have $x \in I_{[\alpha]}$. The reverse implication is clear. Therefore $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$. ∎

Proposition 2.4.7 *For an element $\alpha \in L$, $J_{[\alpha]} = J_{[\alpha^{-1}]}$.*

Proof

$$\begin{aligned}
J_{[\alpha]} &= I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) \\
&= I_{[\alpha]} + I_{[\alpha]}\eta_1 + \cdots + I_{[\alpha]}\eta_d \\
&= \eta_d^{-1}(\eta_d I_{[\alpha]}) + \eta_d^{-1}\eta_1(\eta_d I_{[\alpha]}) + \cdots + (\eta_d I_{[\alpha]}) \\
&= \eta_d^{-1} I_{[\alpha^{-1}]} + \eta_d^{-1}\eta_1 I_{[\alpha^{-1}]} + \cdots + I_{[\alpha^{-1}]} \\
&= I_{[\alpha]}(1, \eta_d^{-1}\eta_{d-1}, \ldots, \eta_d^{-1}\eta_1, \eta_d^{-1}) \\
&= J_{[\alpha^{-1}]}
\end{aligned}$$

∎

Theorem 2.4.8 *It holds that $\alpha \in L$ is super-primitive over R if and only if so is α^{-1}.*

Proof The element α is super-primitive over R if and only if $J_{[\alpha]} \not\subseteq p$ for any $p \in \mathrm{Dp}_1(R)$. Since $J_{[\alpha]} = J_{[\alpha^{-1}]}$ by Proposition 2.4.7, $J_{[\alpha^{-1}]} \not\subseteq p$ for any $p \in \mathrm{Dp}_1(R)$, which means that α^{-1} is super-primitive over R. ∎

Corollary 2.4.9 *Assume that α is anti-integral over R. Then $R[\alpha]$ is flat over R if and only if $R[\alpha^{-1}]$ is flat over R.*

Proof The extension $R[\alpha]$ is flat over R if and only if $J_{[\alpha]} = R$ by Theorem 2.3.6. Since $J_{[\alpha^{-1}]} = J_{[\alpha]} = R$, $R[\alpha^{-1}]$ is flat over R. ∎

2.5 Vanishing Points and Blowing-Up Points

Assume that α is an anti-integral element over R. For $p \in \mathrm{Spec}(R)$, $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p) < \infty$ if and only if $R[\alpha]_p$ is flat over R_p by Theorem 2.2.5. So it may be natural to ask when $\mathrm{rank}_{k(p)} R[\alpha] \otimes_R k(p)$ is infinite or zero.

Let α be an element which is algebraic over R. Recall that $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ is the minimal polynomial of α over K, where $d = [K(\alpha) : K]$ and that $J_{[\alpha]} := I_{[\alpha]}\mathrm{c}(\varphi_\alpha(X)) = I_{[\alpha]} + I_{[\alpha]}\eta_1 + \cdots + I_{[\alpha]}\eta_d$. Let $\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})$, which is an ideal of R.

Recall that $E^{(\alpha)} = \mathrm{Ker}(\pi)$, where $\pi : R[X] \to R[\alpha]$ is an R-algebra homomorphism sending X to α.

We use this notation throughout this section.

Lemma 2.5.1 *Assume that α is an anti-integral element over R. For $q \in$ Spec(R), the following are equivalent:*

(i) $qR[\alpha]_q = R[\alpha]_q$;

(ii) $qR[\alpha] \cap R \not\subseteq q$;

(iii) $q \supseteq \tilde{J}_{[\alpha]}$ *and* $q \not\supseteq I_{[\alpha]}\eta_d$.

Proof (i) $\Rightarrow$ (ii): Since $qR[\alpha]_q = R[\alpha]_q$, there exist $a_i \in q$, $\beta_i \in R[\alpha]$ and $s_i \in R \setminus q$ such that $1 = \sum a_i\beta_i/s_i$. Put $s = \prod s_i$. Then $s = \sum a_i\beta_i b_i \in qR[\alpha] \cap R$ with $s \notin q$, where $s\beta_i/s_i = b_i \in R[\alpha]$. Thus $qR[\alpha] \cap R \not\subseteq q$.
(ii) $\Rightarrow$ (i): Take $s \in qR[\alpha] \cap R$ with $s \notin q$. Then $s \in qR[\alpha]_q$ and s is invertible in $R[\alpha]_q$. Thus $qR[\alpha]_q = R[\alpha]_q$.
(iii) $\Rightarrow$ (ii): Take $a \in I_{[\alpha]}$ with $a\eta_d \notin q$. Put $f(X) = a\varphi_\alpha(X)$ and $a\eta_i = b_i$, $a = b_0$. Then $f(X) = b_0X^d + b_1X^{d-1} + \cdots + b_d$. Since $f(\alpha) = 0$, we have $b_0\alpha^d + b_1\alpha^{d-1} + \cdots + b_d = 0$. Noting that $b_d \notin q$, b_d is a unit in $R[\alpha]_q$. Since $b_0, \ldots, b_{d-1} \in q$, $b_d \in qR[\alpha] \subseteq qR[\alpha]_q$. Thus $qR[\alpha]_q = R[\alpha]_q$.
(ii) $\Rightarrow$ (iii): Since $qR[\alpha]_q = R[\alpha]_q$, $1 = b_0 + b_1\alpha + \cdots + b_n\alpha^n$ for some $b_i \in qR_q$. Put $f(x) = b_nX^n + \cdots + b_1X + b_0 - 1$. Then $f(\alpha) = 0$ and $b_0 - 1$ is a unit in R_q. The kernel of $R_q[X] \longrightarrow R[\alpha]_q$ is $(I_{[\alpha]})_q\varphi_\alpha(X)R_q[X]$. So $f(X) \in (I_{[\alpha]})_q\varphi_\alpha(X)R_q[X]$ and $\mathrm{c}(f(X))_q = R_q$. Thus it follows that $(J_{[\alpha]})_q = (I_{[\alpha]})_q\mathrm{c}(\varphi_\alpha(X))_q = R_q$, which means that $R[\alpha]_q$ is flat over R_q by Theorem 2.2.5. Thus $(I_{[\alpha]})_q\varphi_\alpha(X)R_q[X]$ is an invertible ideal of $R_q[X]$. Hence $(I_{[\alpha]})_q$ is a principal ideal of R_q. Let $(I_{[\alpha]})_q = aR_q$. We shall show that all of $a, a\eta_1, \ldots, a\eta_{d-1}$ belong to qR_q. Note that $f(X) \in a\varphi_\alpha(X)R_q[X]$ because $f(\alpha) = 0$. So there exists $h(X) \in R_q[X]$ such that $f(X) = a\varphi_\alpha(X)h(X)$.

We have $-1 \equiv a\varphi_\alpha(X)h(X) \pmod{qR_q[X]}$. Thus $a\eta_i, a \in qR_q$, for $1 \le i \le d-1$ and $a\eta_d \notin qR_q$. Therefore $I_{[\alpha]}, I_{[\alpha]}\eta_1, \ldots, I_{[\alpha]}\eta_{d-1} \subseteq q$ and $I_{[\alpha]}\eta_d \not\subseteq q$. $\square$

Definition 2.5.2 Let A be an extension of R. We say that a prime ideal p of R is a *vanishing point* of R with respect to A if $pA_p = A_p$.

Recall that A is a *blowing-up at* p or p is a *blowing-up point* of R with respect to A if the following two conditions are satisfied:

(i) $pA_p \cap R_p = pR_p$;

(ii) A_p/pA_p is isomorphic to a polynomial ring $(R_p/pR_p)[T]$.

By Lemma 2.5.1, we obtain the following theorem.

Theorem 2.5.3 *Assume that α is an anti-integral element over R. Then the set of vanishing points of R with respect to A (i.e., $\{q \in Spec(R) \mid qR[\alpha]_q = R[\alpha]_q\}$) is given by $V(\tilde{J}_{[\alpha]}\eta_i) \setminus V(I_{[\alpha]}\eta_d) = \bigcap_{i=0}^{d-1} V(I_{[\alpha]}\eta_i) \setminus V(I_{[\alpha]}\eta_d)$, where $\eta_0 = 1$.*

Proposition 2.5.4 *Assume that α is an anti-integral element of degree d over R. Consider the following conditions:*

(i) *$R[\alpha]$ is flat over R;*

(ii) *$J_{[\alpha]} = R$;*

(iii) *If $pR[\alpha]_p = R[\alpha]_p$ for $p \in Spec(R)$, then $pR[\alpha] = R[\alpha]$.*

Then we have implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). If moreover R is a local ring and $\sqrt{\tilde{J}_{[\alpha]}} \not\supseteq I_{[\alpha]}\eta_d$, then (i), (ii), and (iii) are equivalent to each other.

Proof (i) $\Leftrightarrow$ (ii) was proved in Theorem 2.3.6.
(ii) $\Rightarrow$ (iii): Take $p \in \mathrm{Spec}(R)$ and assume that $pR[\alpha]_p = R[\alpha]_p$. Then $p \supseteq \tilde{J}_{[\alpha]} = I_{[\alpha]} + I_{[\alpha]}\eta_1 + \cdots + I_{[\alpha]}\eta_{d-1}$ and $p \not\supseteq I_{[\alpha]}\eta_d$ by Lemma 2.5.1. Take $a \in I_{[\alpha]}$ and put $f(X) = a\varphi_\alpha(X) = aX^d + a\eta_1 X^{d-1} + \cdots + a\eta_d$. Since $f(\alpha) = 0$, we get $a\eta_d \in pR[\alpha]$ and hence $I_{[\alpha]}\eta_d \subseteq pR[\alpha]$. So $J_{[\alpha]} = \tilde{J}_{[\alpha]} + I_{[\alpha]}\eta_d \subseteq pR[\alpha]$. Since $J_{[\alpha]} = R$, we conclude that $pR[\alpha] = R[\alpha]$. We will show the last part. Since $\sqrt{\tilde{J}_{[\alpha]}} \not\supseteq I_{[\alpha]}\eta_d$, there exists $q \in \mathrm{Spec}(R)$ such that $q \supseteq \tilde{J}_{[\alpha]}$ but $q \not\supseteq I_{[\alpha]}\eta_d$. Thus $qR[\alpha]_q = R[\alpha]_q$ and so $qR[\alpha] = R[\alpha]$. Let m denote the maximal ideal of R. Suppose that $m \supseteq J_{[\alpha]}$. Then we have $R[\alpha]/mR[\alpha] \simeq (R/m)[T]$, a polynomial ring (*cf.* Theorem 2.2.5). Hence $mR[\alpha] \neq R[\alpha]$. But $q \subseteq m$ implies that $mR[\alpha] = R[\alpha]$, a contradiction. Thus we conclude that $J_{[\alpha]} = R$. $\square$

Remark 2.5.5 Let the notation be the same as in Proposition 2.5.4.

(1) When $d = 1$ (i.e., α is an anti-integral element of R), then (i), (ii), and (iii) of Proposition 2.5.4 are equivalent.

(2) $pR[\alpha] \cap R = p$ if and only if there exists $P \in \mathrm{Spec}(R[\alpha])$ such that $P \cap R = p$. ∎

Remark 2.5.6 Let the notation be the same as in Lemma 2.5.1. If $\tilde{J}_{[\alpha]} \subseteq q$, then q is either a vanishing point (i.e., $I_{[\alpha]}\eta_d \not\subseteq q$) or a blowing-up point (i.e., $I_{[\alpha]}\eta_d \subseteq q$). So if $\sqrt{J_{[\alpha]}}$ contains $\sqrt{\tilde{J}_{[\alpha]}}$ properly, there exists a vanishing point. Thus $\mathrm{Spec}(R[\alpha]) \longrightarrow \mathrm{Spec}(R)$ is not surjective. ∎

Proposition 2.5.7 *Assume that α is an anti-integral element of degree d over R. Then $\mathrm{Spec}(R[\alpha]) \longrightarrow \mathrm{Spec}(R)$ is surjective if and only if $\sqrt{J_{[\alpha]}} = \sqrt{\tilde{J}_{[\alpha]}}$.*

Proof ($\Rightarrow$): Since $J_{[\alpha]} \supseteq \tilde{J}_{[\alpha]}$, $\sqrt{J_{[\alpha]}} \supseteq \sqrt{\tilde{J}_{[\alpha]}}$. If $\tilde{J}_{[\alpha]} \subseteq q$ for some $q \in \mathrm{Spec}(R)$, there exists $Q \in \mathrm{Spec}(R[\alpha])$ such that $Q \cap R = q$ because $\mathrm{Spec}(R[\alpha]) \longrightarrow \mathrm{Spec}(R)$ is surjective. So $qR[\alpha]_q \neq R[\alpha]_q$, which means that q is not a vanishing point. Thus by Remark 2.5.6, q is a blowing-up point, that is, $q \supseteq J_{[\alpha]}$. Therefore $\sqrt{J_{[\alpha]}} = \sqrt{\tilde{J}_{[\alpha]}}$.
($\Leftarrow$): Suppose that $\mathrm{Spec}(R[\alpha]) \longrightarrow \mathrm{Spec}(R)$ is not surjective. There exists $q \in \mathrm{Spec}(R)$ such that $qR[\alpha]_q = R[\alpha]_q$. So it follows that $q \supseteq \sqrt{\tilde{J}_{[\alpha]}} = \sqrt{J_{[\alpha]}} \supseteq J_{[\alpha]} \supseteq I_{[\alpha]}\eta_d$, a contradiction. ∎

Proposition 2.5.8 *Let the notation be the same as in Proposition 2.5.7 and let $p \in \mathrm{Spec}(R)$ satisfy $pR[\alpha]_p = R[\alpha]_p$. If $q \supseteq pR[\alpha] \cap R$, then q is a blowing-up point of R with respect to $R[\alpha]$.*

Proof Since $p \in \mathrm{Spec}(R)$ satisfies $pR[\alpha]_p = R[\alpha]_p$, we have $p \supseteq \tilde{J}_{[\alpha]}$ by Lemma 2.5.1. Thus $\eta_d I_{[\alpha]} \subseteq \alpha^d I_{[\alpha]} + \cdots + \eta_{d-1}\alpha I_{[\alpha]} \subseteq \tilde{J}_{[\alpha]}R[\alpha] \subseteq pR[\alpha]$. So $q \supseteq pR[\alpha] \cap R \supseteq \tilde{J}_{[\alpha]} + I_{[\alpha]}\eta_d = J_{[\alpha]}$, which means that q is a blowing-up point. ∎

Remark 2.5.9 Let k be a field, a, b indeterminates, and $R = k[a, b]$. Let α be a root of an equation $aX^2 + bX + a = 0$. Then $J_{[\alpha]} = (a, b)R$ and $\mathrm{grade}((a, b)R) = 2$ so that α is a super-primitive element by Definition 2.2.1. In this case, $J_{[\alpha]} = \tilde{J}_{[\alpha]} = (a, b)R$. Thus $\mathrm{Spec}(R[\alpha]) \longrightarrow \mathrm{Spec}(R)$ is surjective, but not flat. Hence the implication (iii) $\Rightarrow$ (i) in Proposition 2.5.4 does not necessarily hold. ∎

Theorem 2.5.10 *Assume that α is an anti-integral element over R and let $p \in \mathrm{Spec}(R)$. If $R[\alpha]$ is not a blowing-up at q, then depth$(R[\alpha]_Q) =$ depth(R_q) for $Q \in \mathrm{Spec}(R[\alpha])$ with $Q \cap R = q$.*

Proof Since α is an anti-integral element over R and q is not a blowing-up point of R with respect to $R[\alpha]$, $R[\alpha]_q$ is flat over R_q by Corollary 2.2.6. Since $R[\alpha]_Q$ is obtained from $R[\alpha]_q$ by localizing at $QR[\alpha]_q$, $R[\alpha]_Q$ is flat over R_q. So we have $\mathrm{depth}(R_q) \leq \mathrm{depth}(R[\alpha]_Q)$. As q is not a blowing-up point, there exists $a \in I_{[\alpha]}$ such that $a\varphi_\alpha(X)R_q[X] = (E^{(\alpha)})_q$. Put $f(X) := a\varphi_\alpha(X)$. Since $Q \in \mathrm{Spec}(R[\alpha])$, there exists $P \in \mathrm{Spec}(R[X])$ such that $P \supseteq E^{(\alpha)}$ and $Q = P/E^{(\alpha)}$. Then $Q_q = P_q/(E^{(\alpha)})_q = P_q/f(X)R_q[X]$. Since $QR[\alpha]_Q = PR[X]_P/f(X)R[X]_P$, it follows that $\mathrm{depth}(R[\alpha]_Q) = \mathrm{depth}(R[X]_P) - 1$. Now since $P \cap R = q$, we have $P \supseteq qR[X]$. Suppose that $P = qR[X]$. Then $qR[X] = P \supseteq E^{(\alpha)}$, which asserts that q is a blowing-up point. So we have $P \neq qR[X]$. Since $PR_q[X]/qR_q[X](\subseteq k(P)[X]) \neq 0$, we have $PR_q[X] = qR_q[X] + g(X)R_q[X]$ for some $g(X) \in R[X] \setminus qR[X]$. Hence $\mathrm{depth}(R[X]_P) \leq \mathrm{depth}(R[X]_{qR[X]}) + 1$. We obtain that $\mathrm{depth}(R[\alpha]_Q) \leq \mathrm{depth}(R_q)$ because $\mathrm{depth}(R[X]_{qR[X]}) = \mathrm{depth}(R_q)$. Thus $\mathrm{depth}(R_q) = \mathrm{depth}(R[\alpha]_Q)$. ∎

Proposition 2.5.11 *Assume that α is an anti-integral element over R. Then the following statements are equivalent:*

(i) $I_{[\alpha]}R[\alpha] = R[\alpha]$;

(ii) $\eta_d \in R[\alpha]$, $I_{[\alpha]} = I_{\eta_d}$, *and* $J_{\eta_d} = R$.

Proof If $I_{[\alpha]} = R$, then the implications (i) $\Leftrightarrow$ (ii) are trivially valid because $\eta_d \in R$ by the definition of $I_{[\alpha]}$. So we may assume that $I_{[\alpha]} \neq R$.
(i) $\Rightarrow$ (ii): Since $\eta_1, \ldots, \eta_d \in I_{[\alpha]}^{-1}R[\alpha] = I_{[\alpha]}^{-1}I_{[\alpha]}R[\alpha] \subseteq R[\alpha]$, we have $\eta_d \in R[\alpha]$. Put $C := R[\eta_1, \ldots, \eta_d]$. Then $R[\alpha]$ is a free C-module and $R[\alpha]$ is integral over C. So $I_{[\alpha]}R[\alpha] = R[\alpha]$ yields that $I_{[\alpha]}C = C$. Thus C is flat over R. Since $R[\alpha]$ is flat over C, $R[\alpha]$ is flat over R, and hence $J_{[\alpha]} = R$. Now we shall show that $I_{[\alpha]} = I_{\eta_d}$. It is easy to see that $I_{[\alpha]} \subseteq I_{\eta_d}$. Take $p \in \mathrm{Spec}(R)$ with $p \supseteq I_{[\alpha]}$. Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, we have $1 = a_0 + a_1\alpha + \cdots + a_n\alpha^n$, where $a_i \in I_{[\alpha]} \subseteq p$. Since $1 - a_0$ is a unit in R_p, α^{-1} is integral over R. Since α is anti-integral over R, α^{-1} is also anti-integral over R. Thus $\varphi_{\alpha^{-1}}(X) = X^d + (\eta_d^{-1}\eta_{d-1})X^{d-1} + \cdots + (\eta_d^{-1}\eta_1)X + \eta_d^{-1} \in R_p[X]$. So there exists $x \in R_p$ such that $\eta_d = 1/x$, $\eta_i = y_i/x$ with some $y_i \in R_p$. Therefore we have $(I_{[\alpha]})_p = xR_p = (I_{\eta_d})_p$. Since p is arbitrary, we conclude that $I_{[\alpha]} = I_{\eta_d}$. Next we shall show that $J_{\eta_d} = I_{\eta_d}(1, \eta_d) = R$. Suppose that $I_{\eta_d}(1, \eta_d) \subseteq p$ for some $p \in \mathrm{Spec}(R)$. Then $I_{[\alpha]} \subseteq p$. Thus by using the above notation, we have $(I_{\eta_d}(1, \eta_d))_p = x(1, 1/x)R_p = R_p$, which contradicts the assumption $I_{\eta_d}(1, \eta_d) \subseteq p$. Therefore we conclude that $J_{\eta_d} = R$.

(ii) $\Rightarrow$ (i) follows the implications: $R[\alpha] \supseteq I_{[\alpha]}R[\alpha] \supseteq I_{[\alpha]}(1, \eta_d) = I_{\eta_d}(1, \eta_d) = R \ni 1$, and hence $R[\alpha] \supseteq I_{[\alpha]}R[\alpha] \supseteq R[\alpha]$. $\square$

Corollary 2.5.12 *Assume that α is an anti-integral element over R. If $I_{[\alpha]}R[\alpha] = R[\alpha]$, then $R[\alpha] \cap K = R[\eta_d]$.*

Proof Put $C := R[\eta_1, \ldots, \eta_d]$. Then by the proof of Proposition 2.5.11, we have $C \subseteq R[\alpha]$. Since $R[\alpha]$ is a free C-module $C + C\alpha + \cdots + C\alpha^{d-1}$, we conclude that $R[\alpha] \cap K = C$. Besides, as in the proof of Proposition 2.5.11, if $I_{[\alpha]} \subseteq p$, then there exists x in R_p such that $\eta_d = 1/x$ and $\eta_i = y_i/x \in R_p[\eta_d]$ for some $y_i \in R_p$. If $I_{[\alpha]} \not\subseteq p$, then $C_p = R_p = R_p[\eta_d]$. Therefore we have $C = R[\eta_d]$. $\square$

Theorem 2.5.13 *Assume that α is an anti-integral element over R and that $\eta_d \in R[\alpha]$. The following statements are equivalent:*

(i) $I_{[\alpha]}R[\alpha] = R[\alpha]$;

(ii) $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$;

(iii) *either α or α^{-1} is integral over R_p for each $p \in Spec(R)$.*

Proof (i) $\Rightarrow$ (ii): We have $\eta_d \in R[\alpha]$, $I_{[\alpha]} = I_{\eta_d}$, and $J_{\eta_d} = R$ by Proposition 2.5.11. So we obtain $R = J_{\eta_d} = I_{\eta_d}(1, \eta_d) = I_{[\alpha]}(1, \eta_d) = I_{[\alpha]} + \eta_d I_{[\alpha]} = I_{[\alpha]} + I_{[\alpha^{-1}]}$ (*cf.* Lemma 2.4.6).
(ii) $\Rightarrow$ (i): We have $R = I_{[\alpha]} + I_{[\alpha^{-1}]} = I_{[\alpha]} + \eta_d I_{[\alpha]} = I_{[\alpha]}(1, \eta_d)$. Thus $R[\alpha] = I_{[\alpha]}(1, \eta_d)R[\alpha] = I_{[\alpha]}R[\alpha] + I_{[\alpha]}\eta_d R[\alpha] \subseteq I_{[\alpha]}R[\alpha]$ because $\eta_d \in R[\alpha]$. The converse inclusion is obvious. Thus $I_{[\alpha]}R[\alpha] = R[\alpha]$.
(ii) $\Rightarrow$ (iii): Take $p \in \mathrm{Spec}(R)$. Then $I_{[\alpha]} \not\subseteq p$ or $I_{[\alpha^{-1}]} \not\subseteq p$ because $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$. If $I_{[\alpha]} \not\subseteq p$, we have $\eta_1, \ldots, \eta_d \in R_p$. Hence $\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0$, which means that α is integral over R_p. Next, we assume that $I_{[\alpha^{-1}]} \not\subseteq p$. The $\eta_d^{-1}\eta_1, \ldots, \eta_d^{-1}\eta_{d-1}, \eta_d^{-1} \in R_p$. Thus $(\alpha^{-1})^d + (\eta_d^{-1}\eta_{d-1})\alpha^{d-1} + \cdots + \eta_d^{-1}\eta_1\alpha + \eta_d^{-1} = 0$, which means that α^{-1} is integral over R_p.
(iii) $\Rightarrow$ (ii): Suppose that there exists a prime ideal p of R such that $I_{[\alpha]} + I_{[\alpha^{-1}]} \subseteq p$. Note that α or α^{-1} is integral over R_p. Assume that α is integral over R_p. Then α satisfies a monic relation of degree d over R_p because α is anti-integral over R_p. But since $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d \in R_p[X]$, we have $I_{[\alpha]} \not\subseteq p$, a contradiction. Similarly we come to a contradiction when we assume α^{-1} is integral over R_p. $\square$

Proposition 2.5.14 *Assume that α is an anti-integral element of degree d over R. If $\eta_d \in R[\alpha]$, then $\sqrt{I_{[\alpha]}R[\alpha] \cap R} = \sqrt{I_{[\alpha]} + I_{[\alpha^{-1}]}}$.*

Proof Take $p \in \mathrm{Spec}(R)$. Then $I_{[\alpha]}R[\alpha] \cap R \not\subseteq p \Leftrightarrow I_{[\alpha]}R_p[\alpha] = R_p[\alpha] \Leftrightarrow (I_{[\alpha]} + I_{[\alpha^{-1}]})_p = R_p \Leftrightarrow I_{[\alpha]} + I_{[\alpha^{-1}]} \not\subseteq p$. Thus we come to our conclusion by Theorem 2.5.13. $\square$

Chapter 3

Subrings of Anti-Integral Extensions

Let R be a Noetherian domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic field extension L of the quotient field K of R and $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Put $E^{(\alpha)} := \mathrm{Ker}(\pi)$, an ideal of R. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$$

Then η_i $(1 \leq i \leq d)$ are uniquely determined by α. Let $I_{\eta_i} := R :_R \eta_i$ and $\eta_i := \bigcap_{i=1}^{d} I_{\eta_i}$, the latter of which is called a *generalized denominator ideal* of α. We say that α is an anti-integral element if and only if $E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X]$. For $f(X) \in R[X]$, let $\mathsf{c}(f(X))$ denote the ideal of R generated by the coefficients of $f(X)$. For an ideal J of $R[X]$, let $\mathsf{c}(J)$ denote the ideal generated by the coefficients of the elements in J. If α is an anti-integral element, then $\mathsf{c}(E^{(\alpha)}) = \mathsf{c}(I_{[\alpha]}\varphi_\alpha(X)R[X]) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Put $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) | \mathrm{depth}(R_p) = 1\}$, then α is called a *super-primitive* element. It is known that a super-primitive element is an anti-integral element (*cf.* Theorem 2.2.8). By definition, the super-primitive is characterized by the set of $\mathrm{Dp}_1(R)$. Put $R\langle\alpha\rangle := R[\alpha] \cap R[\alpha^{-1}]$.

3.1 Extensions $R[\alpha] \cap R[\alpha^{-1}]$ of Noetherian Domains R

By definition (*cf.* Definition 1.1.4), $R[\alpha] \cap R[\alpha^{-1}] = R$ if α is an anti-integral element of degree 1 over R. So we are interested in the case α is an algebraic element of degree $d > 1$. Therefore we come to study the following problem:

Problem. Let R be a Noetherian integral domain, K denote the quotient field of R, and α be an algebraic element over R of degree d. What is $R[\alpha] \cap R[\alpha^{-1}]$?

In the first part of this section, our results are the following: Let α be an anti-integral element of degree d.

(i) We find a set of generators of $R\langle\alpha\rangle$ as an R-algebra.

(ii) $R\langle\alpha\rangle$ is an R-algebra of finite type.

(iii) For $d = 1, 2, 3$, $R\langle\alpha\rangle$ is completely determined.

(iv) When α is anti-integral over R and $I_{[\alpha]}$ is a radical ideal of R, $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$.

In the second part of this section, we investigate an R-module structure of $R\langle\alpha\rangle$ for an anti-integral element α of degree d over R. The R-module structure of $R\langle\alpha\rangle$ is completely determined.

We start with the following definition.

Definition 3.1.1 Let

$$\Delta(\alpha) := \{a_0\alpha^{d-\ell} + \cdots + a_{d-\ell} \mid a_0X^d + \cdots + a_d \in I_{[\alpha]}\varphi_\alpha(X)R[X],\ 0 \le \ell \le d\}$$

Remark 3.1.2 By definition, it is easy to see the following:

(1) $\Delta(\alpha) \subseteq R\langle\alpha\rangle$;

(2) the R-module generated by $\Delta(\alpha)$ is finitely generated because $I_{[\alpha]}$ is a finitely generated ideal of R. ∎

Lemma 3.1.3 *Assume that α is an anti-integral element of degree d over R. If $a_0\alpha^n + \cdots + a_n = b_0(\alpha^{-1})^m + \cdots + b_m \in R\langle\alpha\rangle$ with $a_i, b_j \in R$, then $b_0 \in I_{[\alpha]}\eta_d$.*

Proof Put $f(X) = a_0X^{n+m} + \cdots + a_nX^m - b_mX^m - \cdots - b_0 \in R[X]$. Then $f(\alpha) = 0$ and hence $f(X) \in \mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Thus $f(0) = -b_0 \in I_{[\alpha]}\eta_d$. ∎

Theorem 3.1.4 *Assume that α is an anti-integral element of degree d. Then $R\langle\alpha\rangle$ is an R-algebra generated by $\Delta(\alpha)$, i.e., $R\langle\alpha\rangle = R[\Delta(\alpha)]$.*

Proof The inclusion $R[\Delta(\alpha)] \subseteq R\langle\alpha\rangle$ is obvious. We must show the converse inclusion. Take $\beta \in R\langle\alpha\rangle$ and put $\beta = a_0\alpha^n + \cdots + a_n = b_0(\alpha^{-1})^m + \cdots + b_m$ with $a_i, b_j \in R$. We shall prove our assertion by induction on m. If $m = 0$, then obviously $\beta \in R \subseteq R[\Delta(\alpha)]$. First, we will show the case $m < d$. By Lemma 3.1.3, we have $b_0 \in I_{[\alpha]}\eta_d$, so that there exist $c_0, \ldots, c_d$

with $c_d = b_0$ such that $c_0\alpha^d + c_1\alpha^{d-1} + \cdots + c_d = 0$. Since $m < d$, putting $\gamma = c_d(\alpha^{-1})^m + \cdots + c_{d-m} = -(c_{d-m-1}\alpha + \cdots + c_0\alpha^{d-m})$, the element $\beta - \gamma$ satisfies our induction hypothesis. Note that $\gamma \in \Delta(\alpha)$ and hence $\beta \in R[\Delta(\alpha)]$. Second, we consider the case $m \geq d$. There exist $c_0, \ldots, c_d$ with $c_d = b_0$ such that $c_0\alpha^d + c_1\alpha^{d-1} + \cdots + c_d = 0$. Put $\gamma = c_d(\alpha^{-1}) = -(c_0\alpha^{d-1} + \cdots + c_{d-1}) \in \Delta(\alpha)$. Then $b_0(\alpha^{-1})^m = b_0(/\alpha)(\alpha^{-1})^{m-1} = c_d(\alpha^{-1})(\alpha^{-1})^{m-1} = \gamma(\alpha^{-1})^{m-1}$. Hence we have $\beta = \gamma(\alpha^{-1})^{m-1} + b_1(\alpha^{-1})^{m-1} + \cdots + b_m = -(c_0\alpha^{d-1} + \cdots + c_{d-1})(\alpha^{-1})^{m-1} + b_1(\alpha^{-1})^{m-1} + \cdots + b_m$, which satisfies the induction hypothesis. Thus $\beta \in R[\Delta(\alpha)]$. ∎

Theorem 3.1.5 *Assume that α is anti-integral over R of degree d. Then $R\langle\alpha\rangle$ is an R-algebra of finite type.*

Proof This follows from Remark 3.1.2(2) and Theorem 3.1.4. ∎

Theorem 3.1.6 *The following statements are equivalent:*

(1) *α is an anti-integral element of degree d over R;*

(2) *$\{a \in R | a\alpha \in R\langle\alpha\rangle\} = I_{[\alpha]}$;*

(3) *each element of $R\langle\alpha\rangle$ is of the form $h(\alpha)$ such that $h(X) \in R[X]$ with* $\deg h(X) \leq d - 1$.

Proof By Lemma 2.4.4 and Theorem 2.4.5, we see that (1) is equivalent to (2).
(1) ⇒ (3) will follow from Theorem 3.1.4 in the sequel.
(3) ⇒ (1): Consider the exact sequence:

$$0 \longrightarrow E^{(\alpha)} \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0$$

Take $f(X) \in E^{(\alpha)}$. Suppose that $\deg f(X) = n$ and $n \geq d$. We shall prove that $f(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$ by induction on n. In the case $n = d$, we can show that

$$I_{[\alpha]}\varphi_\alpha(X)R[X] = \{f(X) \in E^{(\alpha)} | \deg f(X) = d\}$$

So (1) is valid. We will consider the case $n > d$. Put $f(X) = a_nX^n + \cdots + a_1X + a_0$. Since $f(\alpha) = 0$, we have that

$$a_n\alpha^{n-1} + \cdots + a_1 = -a_0\alpha^{-1} \in R[\alpha] \cap R[\alpha^{-1}] = R\langle\alpha\rangle$$

So we see that there exist $b_0, b_1, \ldots, b_{d-1} \in R$ such that $a_n\alpha^{n-1} + \cdots + a_1 = b_{d-1}\alpha^{d-1} + \cdots + b_0$ by the hypothesis. Putting $g(X) = a_nX^{n-1} + \cdots + (a_d -$

$b_{d-1})X^{d-1} + \cdots + (a_1 - b_0)$, we see that $g(\alpha) = 0$ and $\deg g(X) = n - 1$. By the induction hypothesis, $g(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$. Also, since $p(X) = f(X) - Xg(X) \in E^{(\alpha)}$ and $\deg p(X) \leq n - 1$, we have that $p(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$ by induction. Hence $f(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$. Therefore we have proved that α is an anti-integral element of degree d over R. ∎

Theorems 2.4.5 and 3.1.6 yield the following result.

Corollary 3.1.7 *The following statements are equivalent:*

(1) *α is an anti-integral element of degree d over R;*

(2) *α^{-1} is an anti-integral element of degree d over R;*

(3) *$\{a \in R | a\alpha \in R\langle\alpha\rangle\} = I_{[\alpha]}$;*

(4) *$\{a \in R | a(\alpha^{-1}) \in R\langle\alpha\rangle\} = I_{[\alpha^{-1}]}$;*

(5) *each element of $R\langle\alpha\rangle$ is of the form $h(\alpha)$ such that $h(X) \in R[X]$ with $\deg h(X) \leq d - 1$;*

(6) *each element of $R\langle\alpha\rangle$ is of the form $h(\alpha^{-1})$ such that $h(X) \in R[X]$ with $\deg h(X) \leq d - 1$.*

Now we require the following general result for later use.

Theorem 3.1.8 *Let α be an algebraic element over R. Then $R\langle\alpha\rangle$ is integral over R.*

Proof Let V be a valuation ring on the field $K(\alpha)$ which contains R. Then either α or $\alpha^{-1} \in V$. The integral closure of R in $K(\alpha)$ is the intersection of the family of valuation rings on $K(\alpha)$ which contains R by [G,(19.8)]. Hence $R\langle\alpha\rangle$ is integral over R. ∎

Corollary 3.1.9 *Assume that α is anti-integral over R of degree d.*

(i) *If $d = 1$ then $R\langle\alpha\rangle = R$.*

(ii) *If $d = 2$ then $R\langle\alpha\rangle = R[\alpha I_{[\alpha]}]$.*

(iii) *If $d = 3$ then $R\langle\alpha\rangle = R[\alpha I_{[\alpha]}, \alpha^{-1}\eta_3 I_{[\alpha]}]$.*

Proof (i) The case $d = 1$ is shown in Definition 1.1.4 and Remark 2.2.2.
(ii) For the case $d = 2$, $a\alpha^2 + b\alpha + c = 0$ with $a, b, c \in R$. So $a\alpha = -c(\alpha^{-1}) - b$ and hence $a\alpha \in \Delta(\alpha)$. Any element in $\Delta(\alpha)$ is the type $a\alpha$ with $a \in R$. Thus $\Delta(\alpha) \subseteq \alpha I_{[\alpha]}$.

(iii) The case $d = 3$: Since $a\alpha^3 + b\alpha^2 + c\alpha + d = 0$ with $a, b, c, d \in R$, every element of $\Delta(\alpha)$ is either a type $a\alpha^2 + b\alpha + c = -d(\alpha^{-1}) \in \alpha^{-1}\eta_3 I_{[\alpha]}$ or a type $a\alpha = -d(\alpha^{-1})^2 - c(\alpha^{-1}) - b \in \alpha I_{[\alpha]}$. ∎

Lemma 3.1.10 *Assume that α^{-1} is anti-integral over R. Let $g(X) = a_0X^n + a_1X^{n-1} + \cdots + a_n$ be a polynomial in $R[X]$ with $g(\alpha) = 0$. Then $a_0 \in I_{[\alpha]}$.*

Proof Put $Y = 1/X$ and put $f(Y) = (1/X)^n g(X)$. Then $f(Y) \in R[Y]$ and $f(\alpha^{-1}) = 0$. Since α^{-1} is anti-integral over R, $f(Y) = \sum f_i(Y)g_i(Y)$ with $\deg f_i(Y) = d$ and $f_i(\alpha^{-1}) = 0$. In this case, $f(0) = \sum f_i(0)g_i(0)$ and $f_i(0) \in I_{[\alpha]}$. Indeed, put $f_i(Y) = b_d(Y)^d + \cdots + b_0$ with $b_i \in R \Longrightarrow b_d + \cdots + b_0X^d \in R[X] \Longrightarrow b_d + \cdots + b_0\alpha^d = 0 \Longrightarrow b_d/b_0 + (b_{d-1}/b_0)\alpha + \cdots + \alpha^d = 0 \Longrightarrow \eta_i = b_i/b_0 \Longrightarrow b_0 \in \bigcap_{i=1}^d I_{\eta_i} = I_{[\alpha]}$. Therefore the constant term of $(1/X)^n g(X)$ is $f(0) = a_0$, which belongs to $I_{[\alpha]}$. ∎

Theorem 3.1.11 *Assume that α is anti-integral over R and that $I_{[\alpha]}$ is a radical ideal (i.e., $\sqrt{I_{[\alpha]}} = I_{[\alpha]}$) of R. Then $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$.*

Proof Take $\beta \in R[\alpha]$ and write $\beta = a_n\alpha^n + \cdots + a_0$ with $a_i \in R$. We have only to show that if β is integral over R, then $\beta \in R\langle\alpha\rangle$. We prove this assertion by induction on n. The case $n = 0$ is obvious. Let $\beta^\ell + b_1\beta^{\ell-1} + \cdots + b_\ell = 0$ be an integral dependence of β over $R\langle\alpha\rangle$. Since $R\langle\alpha\rangle$ is integral over R by Theorem 3.1.8, we may assume that $b_i \in R$ for all i. Hence $(a_n\alpha^n + \cdots + a_0)^\ell + b_1(a_n\alpha^n + \cdots + a_0)^{\ell-1} + \cdots + b_\ell = 0$. Note that α^{-1} is an anti-integral element of degree d over R by Theorem 2.4.5. By use of Lemma 3.1.10, we therefore obtain $a_n^\ell \in I_{[\alpha]}$. Since $I_{[\alpha]}$ is a radical ideal, we have $a_n \in I_{[\alpha]}$. So there exist $a_n = c_0, c_1, \ldots, c_d \in R$ such that

$$(*) \qquad c_0\alpha^d + c_1\alpha^{d-1} + \cdots + c_d = 0$$

If $n \geq d$, then $\beta = a_n\alpha^n + \cdots + a_0 = c_0\alpha^n + a_{n-1}\alpha^{d-1} + \cdots + a_0 = -(c_1\alpha^{d-1} + \cdots + c_d)\alpha^{n-d} + a_{n-1}\alpha^{n-1} + \cdots + a_0$ and hence $\beta \in R\langle\alpha\rangle$ by induction hypothesis. Next suppose that $n \leq d - 1$. Multiply (*) by $(\alpha^{-1})^{n-d}$ and put $\gamma := c_0\alpha^n + c_1\alpha^{n-1} + \cdots + c_n = -(c_{n+1}(\alpha^{-1}) + \cdots + c_d(\alpha^{-1})^{d-n})$. Then γ is integral over R and belongs to $R\langle\alpha\rangle$. Considering the element $\beta - \gamma$, $c_0 = a_n$ yields $\beta - \gamma \in R\langle\alpha\rangle$ by induction hypothesis. Therefore we conclude that $\beta \in R\langle\alpha\rangle$. ∎

Corollary 3.1.12 *Assume that α is an anti-integral element of degree 1, i.e., $\alpha \in K$. If $I_\alpha := (R :_R \alpha)$ is a radical ideal of R, then R is integrally closed in $R[\alpha]$.*

Proof Since the degree $d = 1$, we have $R\langle\alpha\rangle = R$ (see Remark 2.2.2) and $I_{[\alpha]} = I_\alpha$. Our conclusion follows from Theorem 3.1.11. ∎

Proposition 3.1.13

(i) $I_{[\alpha]} = I_{[\alpha-a]}$ *for every* $a \in R$.

(ii) *If* α *is anti-integral over* R, *then so is* $\alpha - a$ *for every* $a \in R$.

Proof (i) Put $X = Y+a$. Then $\varphi_{\alpha-a}(Y) = (Y+a)^d+\eta_1(Y+a)^{d-1}+\cdots+\eta_d = Y^d + \lambda_1 Y^{d-1} + \cdots + \lambda_d$. Note that $\lambda_i \in R + R\eta_1 + \cdots + R\eta_d$ for all i. Take $x \in I_{[\alpha]}$. Then $x\zeta_i \in R$ for all I. Hence $I_{[\alpha]} \subseteq I_{[\alpha-a]}$. Since $\alpha = (\alpha - a) + a$, we have $I_{[\alpha-a]} \subseteq I_{[\alpha]}$, whence $I_{[\alpha]} = I_{[\alpha-a]}$.
(ii) Let $a \in R$. Let $\tau : R[X] \longrightarrow R[X]$ be an R-algebra homomorphism sending X to $X - a$ and $\pi_a : R[X] \longrightarrow R[\alpha - a] = R[\alpha]$ be an R-algebra homomorphism sending X to $\alpha - a$. Then $\pi_a = \pi \cdot \tau$ and

$$\begin{aligned}
&Ker(\pi_a)\\
&= \tau^{-1}(\mathrm{Ker}(\pi))\\
&= \tau^{-1}(I_{[\alpha]}\varphi_\alpha(X)R[X])\\
&= I_{[\alpha]}\varphi_\alpha(X + a)R[X]\\
&= I_{[\alpha]}\varphi_{\alpha-a}(X)R[X]\\
&= I_{[\alpha-a]}\varphi_{\alpha-a}(X)R[X]
\end{aligned}$$

Thus $\alpha - a$ is anti-integral over R. ∎

Corollary 3.1.14 *If* α *is anti-integral over* R *and if* $I_{[\alpha]}$ *is a radical ideal of* R, *then* $R[\alpha] \cap R[1/(\alpha - a)]$ *is integrally closed in* $R[\alpha]$ *for each* $a \in R$. *In particular,* $R[\alpha] \cap R[1/(\alpha - a)]$ *is the integral closure of* R *in* $R[\alpha]$.

Proof This follows from Theorem 3.1.11 and Proposition 3.1.13. ∎

Proposition 3.1.15 *Let* a *be an element in* R. *Assume that* α *is anti-integral over* R *and that* $I_{[\alpha]}$ *is a radical ideal of* R. *Then*

$$R\langle\alpha\rangle = R[\alpha - a] \cap R[1/(\alpha - a)] = R\langle\alpha - a\rangle$$

Proof Since $I_{[\alpha]}$ is a radical ideal, $R\langle\alpha\rangle$ is the integral closure of R in $R[\alpha]$ by Theorem 3.1.11. By Proposition 3.1.13, $I_{[\alpha]} = I_{[\alpha-a]}$, which is a radical ideal, and α is anti-integral over R. So $R[\alpha]\cap R[1/(\alpha-a)] = R[\alpha-a]\cap R[1/(\alpha-a)]$

is the integral closure of R in $R[\alpha - a] = R[\alpha]$ by Corollary 3.1.14. Thus $R\langle\alpha\rangle = R[\alpha - a] \cap R[1/(\alpha - a)]$. $\square$

Proposition 3.1.15 yields the following corollary.

Corollary 3.1.16 *Assume that α is super-primitive over R and that $I_{[\alpha]}$ is a radical ideal of R. Then*

$$R\langle\alpha\rangle \subseteq R[1/(\alpha - a)]$$

for every $a \in R$.

Theorem 2.4.8 and Proposition 2.2.10 yield the following corollary.

Proposition 3.1.17 *If α is a super-primitive element over R, then so are both $\alpha - a$ and $(\alpha - a)^{-1}$ for every $a \in R$.*

Next we investigate an R-module structure of $R\langle\alpha\rangle$ when α is an anti-integral element of degree d over R. Let $\varphi_i(X) := X^i + \eta_1 X^{i-1} + \eta_2 X^{i-2} + \cdots + \eta_i \in K[X]$ $(1 \le i \le d-1)$, where $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ is the minimal polynomial of α over K, and put $\zeta_i := \varphi_i(\alpha)$ and $\Delta_i := I_{[\alpha]}\zeta_i$ $(1 \le i \le d-1)$, which is an R-module. Under this preparation, we have the following Structure Theorem of $R\langle\alpha\rangle$.

Theorem 3.1.18 *Assume that α is an anti-integral element of degree d over R. Then $R\langle\alpha\rangle = R \oplus \Delta_1 \oplus \cdots \oplus \Delta_{d-1} \cong R \oplus (\bigoplus_{i=1}^{d-1} I_{[\alpha]}\zeta_i)$, as R-modules, where $\bigoplus I_{[\alpha]}$ denotes the direct sum of $d-1$ copies of $I_{[\alpha]}$.*

Proof We shall show that $R\langle\alpha\rangle = R + \Delta_1 + \cdots + \Delta_{d-1}$ as R-modules. The inclusion $R + \Delta_1 + \cdots + \Delta_{d-1} \subseteq R\langle\alpha\rangle$ is obvious. We must show the converse inclusion. Take $\beta \in R\langle\alpha\rangle$ and put $\beta = a_0\alpha^n + \cdots + a_n = b_0(\alpha^{-1})^m + \cdots + b_m$ with $a_i, b_j \in R$. We prove our assertion by induction on m. If $m = 0$, then $\beta = b_0 \in R$. First, we will show the case $m < d$. Put $f(X) = a_0X^{n+m} + \cdots + a_nX^m - b_mX^m - \cdots - b_0 \in R[X]$. Then $f(\alpha) = 0$ and hence $f(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$. Thus $f(0) = -b_0 \in I_{[\alpha]}\eta_d$, so that there exist $c_0, \ldots, c_d \in R$ with $c_d = b_0$ such that $c_0\alpha^d + c_1\alpha^{d-1} + \cdots + c_d = 0$. Indeed, since $b_0 = a\eta_d$ for some $a \in I_{[\alpha]}$, we may put $c_0 = a$, $c_i = a\eta_i$ $(1 \le i \le d)$. Since $m < d$, putting $\gamma = c_d(\alpha^{-1})^m + \cdots + c_{d-m+1} = -(c_{d-m} + c_{d-m-1}\alpha + \cdots + c_0\alpha^{d-m})$, we see that $\gamma \in \Delta_{d-m}$ and the element $\beta - \gamma$ satisfies our induction hypothesis. Since $\gamma \in \Delta_{d-m}$, it follows that $\beta \in R + \Delta_1 + \cdots + \Delta_{d-1}$. Second, we consider the case $m \ge d$. As the above statements, there exist $c_0, \ldots, c_d \in R$ with $c_d = b_0$ such that $c_0\alpha^d + c_1\alpha^{d-1} + \cdots + c_d = 0$. Put $\gamma := c_d(\alpha^{-1}) = -(c_0\alpha^{d-1} + \cdots + c_{d-1}) \in \Delta_{d-1}$. Then $b_0(\alpha^{-1})^m = \gamma(\alpha^{-1})^m$. Hence we have

$\beta = \gamma(\alpha^{-1})^{m-1} + \cdots + b_m = -(c_0\alpha^{d-1} + \cdots + c_{d-1})(\alpha^{-1})^{m-1} + b_1(\alpha^{-1})^{m-1} + \cdots b_m = (b_1 - c_{d-1})(\alpha^{-1})^{m-1} + (b_2 - c_{d-2})(\alpha^{-1})^{m-2} + \cdots + b_m$, which satisfies the induction hypothesis. Thus $\beta \in R + \Delta_1 + \cdots + \Delta_{d-1}$. Hence we have proved that $R\langle\alpha\rangle = R + \Delta_1 + \cdots + \Delta_{d-1}$. Next, we consider the R-homomorphism $\phi_i : I_{[\alpha]} \to \Delta_i$ defined by $\phi_i(a) = a\varphi_i(\alpha)$. This R-homomorphism is an R-isomorphism, that is, $I_{[\alpha]} \cong \Delta_i$ as R-modules. Since $(R\langle\alpha\rangle) \otimes_R K = K(\alpha)$ is a d-dimensional vector space over K, it follows that $R\langle\alpha\rangle$ is an R-module of rank d. Hence $R\langle\alpha\rangle = R \oplus \Delta_1 \oplus \cdots \oplus \Delta_{d-1} \cong R \oplus (\bigoplus I_{[\alpha]})$ as R-modules. $\square$

Combining Theorem 3.1.18 with Corollary 3.1.7, we have the following result.

Proposition 3.1.19 *The following statements are equivalent:*

(1) *α is an anti-integral element of degree d over R;*

(2) *$R\langle\alpha\rangle = R \oplus \Delta_1 \oplus \cdots \oplus \Delta_{d-1}$ as R-modules.*

Proof By Theorem 3.1.6, (2) $\Rightarrow$ (1) and the implication (1) $\Rightarrow$ (2) follows Theorem 3.1.18. $\square$

Proposition 3.1.20 [KY6] *Assume that α is an anti-integral element of degree d over R. Then $R\langle\alpha\rangle$ is a flat R-module if and only if $I_{[\alpha]}$ is an invertible ideal of R. Moreover, in this case, α is a super-primitive element of degree d over R.*

Proof Note that $R\langle\alpha\rangle \cong R \oplus I_{[\alpha]} \oplus \cdots \oplus I_{[\alpha]}$ by Theorem 3.1.18 because α is anti-integral over R. Thus $R\langle\alpha\rangle$ is flat over R if and only if $I_{[\alpha]}$ is invertible. The last part follows from Theorems 2.2.8 and 2.1.8. $\square$

Proposition 3.1.21 *If R is a Noetherian locally factorial domain, then $R\langle\alpha\rangle$ is a flat R-module.*

Proof Since R is a Noetherian locally factorial domain, each divisorial ideal is an invertible ideal ([F, 9.2]). Noting that $I_{[\alpha]} = \bigcap_{i=1}^{d} I_{\eta_i}$ is a divisorial ideal, $I_{[\alpha]}$ is an invertible ideal. Hence $R\langle\alpha\rangle$ is flat R-module. $\square$

Proposition 3.1.22 *Assume that α is an anti-integral element of degree d over R. Then $R\langle\alpha\rangle \cap K = R$.*

Proof The proof is easily seen by Theorem 3.1.18. The case $d = 1$ is seen in [K]. $\square$

In the rest of this section, our objective is to investigate the flatness of a ring extension $R[\alpha] \supseteq R\langle\alpha\rangle$. When α is an anti-integral element over $R\langle\alpha\rangle$, we give an equivalent condition for an extension $R[\alpha] \supseteq R\langle\alpha\rangle$ to be flat in terms of the ideal $I_{[\alpha]} + I_{[\alpha^{-1}]}$. From this result, we give a property of an extension $R[\alpha] \supseteq R$ with respect to the flatness when α is a super-primitive element over R. Also, when α is an anti-integral element over R, we give an equivalent condition for the element α to be super-primitive over $R\langle\alpha\rangle$ using the ideal $I_{[\alpha]}+I_{[\alpha^{-1}]}$.

In what follows, we fix the following notation unless otherwise specified:

$$\begin{aligned} I_\alpha^{R\langle\alpha\rangle} &:= \{b \in R\langle\alpha\rangle | b\alpha \in R\langle\alpha\rangle\}, \\ I_\alpha &:= R :_R \alpha R \ for \alpha \in K and J_\alpha := I_\alpha + \alpha I_\alpha, \\ \zeta_i &:= \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i = \varphi_i(\alpha). \end{aligned}$$

Theorem 3.1.23 *Assume that α is an anti-integral element of degree d over R. Then $I_\alpha^{R\langle\alpha\rangle} = I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$.*

Proof From Theorem 3.1.19, we have $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$. Take an element $\beta \in I_\alpha^{R\langle\alpha\rangle}$. Since $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$, we may put $\beta = a + b_1\zeta_1 + \cdots + b_{d-1}\zeta_{d-1}$ with $a \in R$ and $b_i \in I_{[\alpha]}$ $(1 \le i \le d-1)$. It follows that $a\alpha \in R\langle\alpha\rangle$ since $\beta\alpha \in R\langle\alpha\rangle$. Thus $a \in R\langle\alpha\rangle :_R \alpha = I_{[\alpha]}$ by Theorem 3.1.6. Therefore $I_\alpha^{R\langle\alpha\rangle} \subseteq I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$. We shall show the converse inclusion. Take an element $a \in I_{[\alpha]}$. Since $a\eta_i \in R$ and $a\zeta_1 \in I_{[\alpha]}\zeta_1$, we have $a\alpha = a(\zeta_1 - \eta_1) = a\zeta_1 - a\eta_1 \in R\langle\alpha\rangle$. Hence $I_{[\alpha]} \subseteq I_\alpha^{R\langle\alpha\rangle}$. Furthermore for any $a \in I_{[\alpha]}$, we have $\alpha(a\zeta_{d-i}) = a\alpha(\alpha^{d-i} + \eta_1\alpha^{d-i-1} + \cdots + \eta_{d-1}) = -a\alpha\{\eta_{d-i+1}\alpha^{-1} + \cdots + \eta_d(\alpha^{-1})^i\} = -a\{\eta_{d-i+1} + \eta_{d-i+2}\alpha^{-1} + \cdots + \eta_d(\alpha^{-1})^{i-1}\} \in R\langle\alpha\rangle$. Thus $I_\alpha^{R\langle\alpha\rangle} \supseteq I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$. ∎

Remark 3.1.24 Since $R\langle\alpha\rangle \subseteq R\langle\alpha\rangle[\alpha] \cap R\langle\alpha\rangle[\alpha^{-1}] \subseteq R\langle\alpha\rangle$, we see that α is an anti-integral element over $R\langle\alpha\rangle$. ∎

Proposition 3.1.25 *Assume that α is an anti-integral element of degree d over R. Then the following assertions hold.*

(1) *if p is a prime ideal of R containing $I_{[\alpha]}$, then $P := p \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ is a prime ideal of $R\langle\alpha\rangle$.*

(2) *if p is a prime divisor of $I_{[\alpha]}$, then P is a prime ideal of $R\langle\alpha\rangle$ with depth 1.*

Proof (1) By Theorem 3.1.23, $I_\alpha^{R\langle\alpha\rangle} = I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ is an ideal of $R\langle\alpha\rangle$. Thus $(I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1})R\langle\alpha\rangle \subseteq I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$.

So $PR\langle\alpha\rangle = pR\langle\alpha\rangle \oplus (I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1})R\langle\alpha\rangle \subseteq p \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1} = P$. Therefore P is an ideal of $R\langle\alpha\rangle$. Also, since $R\langle\alpha\rangle/P = R/p$ is an integral domain, P is a prime ideal of $R\langle\alpha\rangle$.
(2) Since p is a prime divisor of $I_{[\alpha]}$, there exists an element a of R such that $p = I_{[\alpha]} :_R a$. Now we shall show that $P = I_\alpha^{R\langle\alpha\rangle} :_{R\langle\alpha\rangle} a$. Indeed, for any $\beta \in P$, we can write $\beta = x + b_1\zeta_1 + \cdots + b_{d-1}\zeta_{d-1}$, where $x \in p$ and $b_i \in I_{[\alpha]}$ $(1 \leq i \leq d-1)$. We have only to prove that $x \in I_{[\alpha]} :_R a = p$. Since $a\beta \in I_\alpha^{R\langle\alpha\rangle}$, it follows that $ax \in I_{[\alpha]}$. Thus $x \in I_{[\alpha]} :_R a = p$ and so $\beta \in P$. Hence $P = I_\alpha^{R\langle\alpha\rangle} :_{R\alpha\rangle} a$. Since P is a prime divisor of $I_\alpha^{R\langle\alpha\rangle}$ and $I_\alpha^{R\alpha\rangle}$ is a divisorial ideal, we conclude that $P \in \mathrm{Dp}_1(R\langle\alpha\rangle)$ by [Y, Proposition 1.10]. $\square$

Proposition 3.1.26 *Assume that α is an anti-integral element of degree d over R. Then a prime ideal P of $R\langle\alpha\rangle$ satisfies one of the following three statements. Put $p := P \cap R$.*

(1) *The case that $p \not\supseteq I_{[\alpha]}$. Then $R\langle\alpha\rangle_p = R_p[\alpha]$ and $R_p[\alpha]$ is a free R_p-module of rank d. Furthermore P is a prime ideal of $R\langle\alpha\rangle$ with depth 1 if and only if p is a prime ideal of R with depth 1.*
(2) *The case that $p \supseteq I_{[\alpha]}$ but $P \not\supseteq I_\alpha^{R\langle\alpha\rangle}$. Then $R[\alpha] \subseteq R\langle\alpha\rangle_P$ and $R\langle\alpha\rangle_P = R[\alpha]_{\overline{p}}$, where $\overline{p} = PR\langle\alpha\rangle_P \cap R[\alpha]$.*
(3) *The case that $P \supseteq I_\alpha^{R\langle\alpha\rangle}$. Then $p \supseteq I_{[\alpha]}$ and $P = p \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$. In particular, if $P \in \mathrm{Ass}_{R\langle\alpha\rangle}(R[\alpha]/R\langle\alpha\rangle)$, then P is a prime ideal of depth 1.*

Proof (1) Since $p \not\supseteq I_{[\alpha]}$, we see that $(I_{[\alpha]})_p = R_p$. So it is clear that $R\langle\alpha\rangle_p = R_p \oplus R_p\zeta_1 \oplus \cdots \oplus R_p\zeta_{d-1}$. Thus $R\langle\alpha\rangle_p$ is a free R_p-module of rank d. Take $a \in I_{[\alpha]} \setminus p$. Then $(a\alpha)^d + (a\eta_1)(a\alpha)^{d-1} + \cdots a^d\eta_d = 0$. Hence $a\alpha$ is integral over R and so $a\alpha$ is integral over R_p. Since $R\langle\alpha\rangle$ is integral over R, we conclude that $R_p \subseteq R_p[\alpha] \subseteq R_p[\alpha^{-1}]$. Thus we have $R_p[\alpha] = R_p[\alpha] \cap R_p[\alpha^{-1}] = R\langle\alpha\rangle_p$. Since $R\langle\alpha\rangle_p$ is flat over R_p, it follows that $\mathrm{depth}(R\langle\alpha\rangle_p) = \mathrm{depth}(R_p)$. In particular, $P \in \mathrm{Dp}_1(R\langle\alpha\rangle)$ if and only if $p \in \mathrm{Dp}_1(R)$.
(2) Since $P \not\supseteq I_\alpha^{R\langle\alpha\rangle}$ and $\alpha \in R\langle\alpha\rangle_P$, it follows that $R[\alpha] \subseteq R\langle\alpha\rangle_P$. Hence we have $R\langle\alpha\rangle_P = R[\alpha]_{\overline{p}}$.
(3) Put $P' = p \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$. Then P' is a prime ideal of $R\langle\alpha\rangle$ by Proposition 3.1.25(1). It is obvious that $P, P' \supseteq I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$. Thus $P \cap R = P' \cap R = p$ and so $P = P'$. If $P \in \mathrm{Ass}_{R\langle\alpha\rangle}(R[\alpha]/R\langle\alpha\rangle)$, then $P = R\langle\alpha\rangle :_{R\langle\alpha\rangle} x$ for some $x \in R[\alpha] \setminus R\langle\alpha\rangle$. Since P is a divisorial ideal, we see that P is an ideal of depth 1 in $R\langle\alpha\rangle$ by [Y, Proposition 1.10]. Suppose that $P \not\supseteq I_\alpha^{R\langle\alpha\rangle}$. In the same way as in (2), we can prove that $R[\alpha] \subseteq$

$R\langle\alpha\rangle_P$ and hence $R\langle\alpha\rangle_P = R[\alpha]_{\overline{p}}$, which contradicts the assumption that $P \in \mathrm{Ass}_{R\langle\alpha\rangle}(R[\alpha]/R\langle\alpha\rangle)$. Thus $p \supseteq I_\alpha^{R\langle\alpha\rangle}$ and so $p \supseteq I_\alpha^{R\langle\alpha\rangle} \cap R = I_{[\alpha]}$. ∎

We obtain the following theorem from the arguments above.

Theorem 3.1.27 *Assume that α is an anti-integral element of degree d over R. Then the following two conditions are equivalent to each other:*

(1) *$R[\alpha]$ is a flat R-module;*

(2) *$R\langle\alpha\rangle[\alpha] = R[\alpha]$ is a flat $R\langle\alpha\rangle$-module.*

Proof (1) $\Rightarrow$ (2): We must show that $R\langle\alpha\rangle[\alpha]$ is flat over $R\langle\alpha\rangle$. For this end, we have only to show that $J_\alpha^{R\langle\alpha\rangle} = R\langle\alpha\rangle$. Since $J_\alpha^{R\langle\alpha\rangle} = I_\alpha^{R\langle\alpha\rangle} + \alpha I_\alpha^{R\langle\alpha\rangle} = I_\alpha^{R\langle\alpha\rangle} + I_{\alpha^{-1}}^{R\langle\alpha\rangle}$, it is enough to prove that $I_\alpha^{R\langle\alpha\rangle} + I_{\alpha^{-1}}^{R\langle\alpha\rangle} = R\langle\alpha\rangle$. Suppose that $I_\alpha^{R\alpha\rangle} + I_{\alpha^{-1}}^{R\langle\alpha\rangle} \subseteq P$ for some $P \in \mathrm{Spec}(R\langle\alpha\rangle)$. By Proposition 3.1.25, we have that $P = p \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$. On the other hand, since $I_\alpha^{R\langle\alpha\rangle} = I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ by Theorem 3.1.23, we conclude that $I_{[\alpha]} \subseteq p$. Note that $I_{\alpha^{-1}}^{R\langle\alpha\rangle} = I_{[\alpha^{-1}]} \oplus I_{[\alpha^{-1}]}\zeta_1' \oplus \cdots \oplus I_{[\alpha^{-1}]}\zeta_{d-1}' \subseteq P$, where $\zeta_i' = -\eta_d^{-1}\zeta_{d-1} + \eta_d^{-1}\eta_{d-i}$. Take an element $a \in I_{[\alpha]}$. Then $a\eta_d\zeta_{d-i}' = a(-\zeta_{d-i} + \eta_{d-i}) = -a\eta_d^{-1}\zeta_{d-i} + a\zeta_{d-i}$. Since $a\zeta_{d-i} \in P$, we get $a\eta_{d-i} \in P \cap R = p$ for all i $(1 \le i \le d-1)$. Also we see that $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]} \subseteq P \cap R = p$ and hence that $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) \subseteq p$. Since $R[\alpha]$ is a flat extension of R, we have $J_{[\alpha]} = R$ by Theorem 2.3.6, a contradiction. Thus $R\langle\alpha\rangle[\alpha]$ is a flat $R\langle\alpha\rangle$-module.
(2) $\Rightarrow$ (1): By Remark 3.1.25, the element α is an anti-integral element over $R\langle\alpha\rangle$. Since $R\langle\alpha\rangle[\alpha]$ is a flat $R\langle\alpha\rangle$-module, we see that $R\langle\alpha\rangle = J_\alpha^{R\langle\alpha\rangle} = I_\alpha^{R\langle\alpha\rangle} + \alpha I_\alpha^{R\langle\alpha\rangle} = I_\alpha^{R\langle\alpha\rangle} + I_{\alpha^{-1}}^{R\langle\alpha\rangle}$ from Theorem 1.2.8. Note that $I_\alpha^{R\langle\alpha\rangle} = I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ and $I_{\alpha^{-1}}^{R\langle\alpha\rangle} = I_{[\alpha^{-1}]} \oplus I_{[\alpha^{-1}]}\zeta_1' \oplus \cdots \oplus I_{[\alpha^{-1}]}\zeta_{d-1}'$, where $\zeta_i' = -\eta_d^{-1}\zeta_{d-1} + \eta_d^{-1}\eta_{d-i}$. It is easy to see that $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$, $\alpha I_\alpha^{R\langle\alpha\rangle} = I_{\alpha^{-1}}^{R\langle\alpha\rangle}$ and that $R\langle\alpha\rangle[\alpha]/R\langle\alpha\rangle$ is a flat extension. We obtain that $R\langle\alpha\rangle = I_\alpha^{R\langle\alpha\rangle} + I_{\alpha^{-1}}^{R\langle\alpha\rangle)} \subseteq I_{[\alpha]} \oplus I_{[\alpha^{-1}]} \oplus I_{[\alpha]}(\eta_1, \ldots, \eta_{d-1}) \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) \oplus I_{[\alpha]}\zeta_1 \oplus \cdots I_{[\alpha]}\zeta_{d-1}$. Since $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$, we see that $R = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = J_{[\alpha]}$. From Theorem 2.3.6, it follows that $R[\alpha]$ is a flat extension of R. ∎

Lemma 3.1.28 *Assume that α is an anti-integral element of degree d over R. Then $P \cap R =: p \in \mathrm{Dp}_1(R)$ for every $P \in \mathrm{Dp}_1(R\langle\alpha\rangle)$.*

Proof Let $P \in \mathrm{Dp}_1(R\langle\alpha\rangle)$ and put $P \cap R = p$. Then there exists an element β of $K[\alpha]$ such that $P = R\langle\alpha\rangle :_{R\langle\alpha\rangle} \beta$ by [Y, Proposition 1.10]. We have $K[\alpha] = K \oplus K\zeta_1 \oplus K\zeta_2 \oplus \cdots \oplus K\zeta_{d-1}$ and $R\langle\alpha\rangle = I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus I_{[\alpha]}\zeta_2 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$.

We write $\beta = \lambda_0 + \lambda_1\zeta_1 + \cdots + \lambda_{d-1}\zeta_{d-1}$, where $\lambda_i \in K$ for all i $(1 \le i \le d-1)$. Thus we conclude that $p = P \cap R = (R :_R \lambda_0) \cap (\bigcap_{i=1}^{d-1}(I_{[\alpha]} :_R \lambda_i))$. Since $I_{[\alpha]}$ is a divisorial ideal, we get $p \in \mathrm{Dp}_1(R)$ by [Y, Proposition 1.10]. ∎

Theorem 3.1.29 *Assume that α is an anti-integral element of degree d over R. Then the following two conditions are equivalent:*

(1) *α is a super-primitive element over $R\langle\alpha\rangle$;*

(2) $\mathrm{grade}_R(I_{[\alpha]} + I_{[\alpha^{-1}]}) > 1$.

Proof (1) $\Rightarrow$ (2): Suppose that $I_{[\alpha]} + I_{[\alpha^{-1}]} \subseteq p$ for some $p \in \mathrm{Dp}_1(R)$. Since α is super-primitive over $R\langle\alpha\rangle$, we have $(I_{[\alpha]})_p = aR_p$ for some $a \in R$ by Theorem 2.3.10. Thus pR_p is a prime divisor of aR_p. Thus p is a prime divisor of $I_{[\alpha]}$. Hence $P = p \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1} \in \mathrm{Dp}_1(R\langle\alpha\rangle)$ by Proposition 3.1.25. Since $p \supseteq I_{[\alpha]}$, we have that $P \supseteq I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1} = I_\alpha^{R\langle\alpha\rangle}$. Similarly, we have $P \supseteq I_{\alpha^{-1}}^{R\langle\alpha\rangle} = I_{[\alpha^{-1}]} \oplus I_{[\alpha^{-1}]}\zeta_1' \oplus \cdots \oplus I_{[\alpha^{-1}]}\zeta_{d-1}'$ because $p \supseteq I_{[\alpha^{-1}]}$, where ζ_i' is the same as in the proof of Theorem 3.1.28. Thus we get $P \supseteq I_{[\alpha]} + I_{[\alpha^{-1}]} = J_\alpha^{R\langle\alpha\rangle}$, which contradicts the assumption that α is a super-primitive element over $R\langle\alpha\rangle$. Hence we conclude that $\mathrm{grade}_R(I_{[\alpha]} + I_{[\alpha^{-1}]}) > 1$.
(2) $\Rightarrow$ (1): Suppose that α is not a super-primitive element over $R\langle\alpha\rangle$. Then there exists a prime ideal P of $R\langle\alpha\rangle$ such that $\mathrm{depth}(R\langle\alpha\rangle_P) = 1$ and that $P \supseteq J_\alpha^{R\langle\alpha\rangle} = I_\alpha^{R\langle\alpha\rangle} + \alpha I_\alpha^{R\langle\alpha\rangle} = I_\alpha^{R\langle\alpha\rangle} + I_{\alpha^{-1}}^{R\langle\alpha\rangle}$. From Lemma 3.1.23, it follows that $I_\alpha^{R\langle\alpha\rangle} = I_{[\alpha]} \oplus I_{[\alpha]}\zeta_1 \oplus I_{[\alpha]}\zeta_2 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$. So we have $I_\alpha^{R\langle\alpha\rangle} \cap R = I_{[\alpha]}$. Similarly, we get $I_{\alpha^{-1}}^{R\langle\alpha\rangle} \cap R = I_{[\alpha^{-1}]}$. Thus we have $P \cap R \supseteq I_\alpha^{R\langle\alpha\rangle} \cap R = I_{[\alpha]}$ and $P \cap R \supseteq I_{\alpha^{-1}}^{R\langle\alpha\rangle} \cap R = I_{[\alpha^{-1}]}$. Hence $P \cap R \supseteq I_{[\alpha]} + I_{[\alpha^{-1}]}$. By Lemma 3.1.29, we see that $P \cap R \in \mathrm{Dp}_1(R)$, which contradicts the assumption that $\mathrm{grade}_R(I_{[\alpha]} + I_{[\alpha^{-1}]}) > 1$. Consequently, α is a super-primitive element over $R\langle\alpha\rangle$. ∎

For an ideal I, recall that $\mathcal{R}(I) := I :_K I$, which is an over-ring of R.

Definition 3.1.30 For $\alpha \in L$, let

$$D_{(\alpha)} := R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1} \qquad \text{(as } R\text{-modules)}$$

where $\zeta_i := \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i$ for i $(1 \le i \le d-1)$.

Remark 3.1.31 $R\langle\alpha\rangle$, $\mathcal{R}(R)$, and $D_{(\alpha)}$ are integral over R. ∎

The proof of the following lemma is obtained from the proof of Theorem 3.1.18 without the assumption that α is anti-integral over R.

Lemma 3.1.32 *Under the same situation as in Definition 3.1.30, $D_{(\alpha)}$ is a subring of $R\langle\alpha\rangle$ and they are birational i.e., $D_{(\alpha)}$ and $R\langle\alpha\rangle$ have the same quotient fields $K(\alpha)$.*

Proof When $d = 1$, $D_{(\alpha)} = R$, we are done.

Assume that $d = 2$. Note $\zeta_1 = \alpha + \eta_1$ and hence $\zeta_1^2 = \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1 = \eta_1\zeta_1 - \eta_2$ (here $\zeta_2 = 0$).

Assume that $d = 3$. Note $\zeta_1 = \alpha + \eta_1$, $\zeta_2 = \alpha\zeta_1 + \eta_2$, and $\zeta_3 = \alpha\zeta_2 + \eta_3$. Thus

$$\zeta_1^2 = \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1,$$

$$\zeta_2\zeta_1 = \zeta_2(\alpha + \eta_1) = \alpha\zeta_2 + \zeta_2\eta_1 = \zeta_3 - \eta_3 + \eta_1\zeta_2 = -\eta_3 + \eta_1\zeta_2, \text{ and}$$

$$\zeta_2^2 = \zeta_2(\alpha\zeta_1 + \eta_2) = (\alpha\zeta_2)\zeta_1 + \zeta_2\eta_2 = (\zeta_3 - \eta_3)\zeta_1 + \zeta_2\eta_2 = -\eta_3\zeta_1 + \eta_2\zeta_2$$

because $\zeta_3 = 0$.

Assume that $d \geq 3$. Note first that $\eta_0 := 1$, $\zeta_0 := 1$, $\zeta_d = 0$ and $\zeta_{i+1} = \alpha\zeta_i + \eta_{i+1}$. Now we compute $\zeta_i\zeta_j$ as follows:

$$\begin{aligned}
\zeta_i\zeta_j &= \zeta_i(\alpha\zeta_{j-1} + \eta_j) \\
&= \alpha\zeta_i\zeta_{j-1} + \eta_j\zeta_i \\
&= (\zeta_{i+1} - \eta_{i+1})\zeta_{j-1} + \eta_j\zeta_i \\
&= \zeta_{i+1}\zeta_{j-1} - \eta_{i+1}\zeta_{j-1} + \eta_j\zeta_i \\
&= \zeta_{i+2}\zeta_{i-2} - \eta_{i+2}\zeta_{j-2} + \eta_{j-1}\zeta_{i+1} - \eta_{i+1}\zeta_{j-1} + \eta_j\zeta_i \\
&= \zeta_{i+2}\zeta_{i-2} - (\eta_{i+2}\zeta_{j-2} + \eta_{i+1}\zeta_{j-1}) + (\eta_j\zeta_i + \eta_{j-1}\zeta_{i+1}) \\
&\quad \cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots
\end{aligned}$$

(i) Repeat the above process, we have

$$\zeta_i\zeta_j = \zeta_{i+j}\zeta_0 - \sum_{t=1}^{j} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s}$$

that is,

$$\zeta_i\zeta_j = \zeta_{i+j} - \sum_{t=1}^{j} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s}$$

(ii) Put $\ell := i + j - d$. Then $j \leq i < d$ yields $\ell < d - 1$ and $j - \ell \geq 1$. Thus continuing the above process yields

$$\zeta_i\zeta_j = \zeta_d\zeta_\ell - \sum_{t=1}^{j-\ell} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-\ell-1} \eta_{j-s}\zeta_{i+s}$$

that is,

$$\zeta_i\zeta_j = -\sum_{t=1}^{d-i}\eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{d-i-1}\eta_{j-s}\zeta_{i+s}$$

Thus we have $I_{[\alpha]}\zeta_i I_{[\alpha]}\zeta_j = I_{[\alpha]}^2\zeta_i\zeta_j \in D_{(\alpha)}$. Hence for $\beta, \gamma \in D_{(\alpha)}$, we have $\beta+\gamma, \beta\gamma \in D_{(\alpha)}$, which shows that $D_{(\alpha)}$ is a subring of $R[\alpha]$.

Next, it is obvious that $D_{(\alpha)} \subseteq R\langle\alpha\rangle$. Take $\zeta_1 = \alpha + \eta_1 \in D_{(\alpha)}$. Then $\alpha = \zeta_1 - \eta_1 \in K(D_{(\alpha)})$. So we have $K(\alpha) = K(D_{(\alpha)}) \subseteq K(R\langle\alpha\rangle) = K(\alpha)$. $\square$

Definition 3.1.33 For $\alpha \in L$, define:

$$\mathrm{Ant}(\alpha) := \{c \in R \setminus (0) \mid R\langle\alpha\rangle[1/c] = D_{(\alpha)}[1/c]\} \cup \{0\}$$

and

$$\mathrm{Sup}(\alpha) := \{c \in R \setminus (0) \mid \mathcal{R}(I_{[\alpha]})[1/c] = R[1/c]\} \cup \{0\}$$

The set $\mathrm{Ant}(\alpha)$ is called an *obstruction of anti-integrality* of α over R and the set $\mathrm{Sup}(\alpha)$ is called an *obstruction of super-primitiveness* of α over R.

Theorem 3.1.34 *For $\alpha \in L$,*

(1) $\mathrm{Ant}(\alpha)$ *is a radical ideal of R which contains the ideal $I_{[\alpha]}$;*

(2) *for $p \in \mathrm{Spec}(R)$, $\mathrm{Ant}(\alpha) \not\subseteq p$ if and only if α is an anti-integral element over R_p;*

(3) *the following conditions are equivalent:*

(3.i) *α is anti-integral over R,*

(3.ii) $\mathrm{Ant}(\alpha) = R$,

(3.iii) $\mathrm{Ant}(\alpha) \not\subseteq p$ *for all* $p \in \mathrm{Spec}(R)$,

(3.iv) $\mathrm{Ant}(\alpha) \not\subseteq p$ *for all* $p \in \mathrm{Dp}_1(R)$.

Proof (1) First, take $a, b \in \mathrm{Ant}(\alpha)$ and put $c := a + b$. Take $x \in R\langle\alpha\rangle$. Then $R\langle\alpha\rangle[1/a] = D_{(\alpha)}[1/a]$ and $R\langle\alpha\rangle[1/b] = D_{(\alpha)}[1/b]$. So there exists $n \in \mathbf{N}$ such that $a^n x \in D_{(\alpha)}$ and $b^n x \in D_{(\alpha)}$. Thus we have $c^{2n}x = (a+b)^{2n} = \sum_{i+j=2n} a^i b^j x = \sum_{i+j=2n, i\geq n} a^i b^j x + \sum_{i+j=2n, j\geq n} a^i b^j x \in D_{(\alpha)} + D_{(\alpha)} = D_{(\alpha)}$, which implies that $c^{2n}x \in D_{(\alpha)}$ and hence $x \in D_{(\alpha)}[1/c]$. Therefore $\mathrm{Ant}(\alpha)$ is an ideal of R. Second, since $R\langle\alpha\rangle[1/e] = R\langle\alpha\rangle[1/e^n]$ and $D_{(\alpha)}[1/e] = D_{(\alpha)}[1/e^n]$ for every $n \in \mathbf{N}$. So $\mathrm{Ant}(\alpha)$ is a radical ideal. Third, take a nonzero $a \in I_{[\alpha]}$. Then $I_{[\alpha]}[1/a] = R[1/a]$. Hence $\varphi_\alpha(X) \in R[1/a][X]$, that is, $\eta_i \in R[1/a]$ for all i $(1 \leq i \leq d)$. Since $\deg(\varphi_\alpha(X)) = d$, α

is integral over $R[1/a]$. Thus $\alpha \in R[1/a][\eta_1, \ldots, \eta_{d-1}] = D_{(\alpha)}[1/a]$. So we have $R\langle\alpha\rangle[1/a] \subseteq R[\alpha][1/a] \subseteq D_{(\alpha)}[1/a] \subseteq R\langle\alpha\rangle[1/a]$ and hence $R\langle\alpha\rangle[1/a] = D_{(\alpha)}[1/a]$, which shows that $a \in \mathrm{Ant}(\alpha)$. Therefore we have $I_{[\alpha]} \subseteq \mathrm{Ant}(\alpha)$.
(2) Let $p \in \mathrm{Spec}(R)$. Assume that $\mathrm{Ant}(\alpha) \not\subseteq p$. Then there exists $a \in \mathrm{Ant}(\alpha) \setminus p$ such that $R\langle\alpha\rangle[1/a] = D_{(\alpha)}[1/a]$. Then $R\langle\alpha\rangle_p = (D_{(\alpha)})_p$. Take $f(X) \in \mathrm{Ker}(\pi)$. with $n := \deg(f(X))(\geq d)$. Put $f(X) := a_0X^n + a_1X^{n-1} + \cdots + a_n \in R[X]$. Then $a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0$ and hence $a_0\alpha^{d-1} + a_1\alpha^{d-2} + \cdots + a_{d-1} = -(a_d(1/\alpha) + \cdots + a_n(1/\alpha)^{n-d}) \in R\langle\alpha\rangle_p = R_p\langle\alpha\rangle = (D_{(\alpha)})_p = R_p \oplus (I_{[\alpha]})_p\zeta_1 \oplus \cdots \oplus (I_{[\alpha]})_p\zeta_{d-1}$. Thus we have $a_0 \in (I_{[\alpha]})_p$. Consider $f_2(X) := f(X) - a_0\varphi_\alpha(X)X^{n-d}$. Then $\deg(f_2(X)) \leq n-1$ and $f_2(X) \in \mathrm{Ker}(\pi)$. By induction on n, we have $(\mathrm{Ker}(\pi))_p \subseteq I_{[\alpha]}\varphi_\alpha(X)R_p[X]$. Hence $\mathrm{Ker}(\pi)_p = I_{[\alpha]}\varphi_\alpha(X)R_p[X]$, which shows that α is anti-integral over R_p. Conversely, assume that α is anti-integral over R_p. Then we have $R_p\langle\alpha\rangle = (D_{(\alpha)})_p$ by Theorem 3.1.18. Since $(D_{(\alpha)})_p$ is a finitely generated R_p-module, so is $R_p\langle\alpha\rangle$. Thus there exists $a \in R \setminus (0)$ such that $R\langle\alpha\rangle[1/a] = D_{(\alpha)}[1/a]$.
(3)(3.iii) $\Leftrightarrow$ (3.ii) is obvious. (3.iv) $\Leftrightarrow$ (3.ii) follows from Theorem 2.4.3.
(3.i) $\Leftrightarrow$ (3.ii): Since α is anti-integral over R_p for every $p \in \mathrm{Spec}(R)$, we have $\mathrm{Ant}(\alpha)_p = R_p$ by (2). Thus $\mathrm{Ant}(\alpha) = R$. The converse implication is obvious. ∎

Theorem 3.1.35 *For $\alpha \in L$,*

(1) $\mathrm{Sup}(\alpha)$ *is a radical ideal of R which contains the ideal $I_{[\alpha]}$;*

(2) *for $p \in \mathrm{Spec}(R)$, $\mathrm{Sup}(\alpha) \not\subseteq p$ if and only if α is a super-primitive element over R_p;*

(3) *The following conditions are equivalent:*

(3.i) *α is super-primitive over R,*

(3.ii) $\mathrm{Sup}(\alpha) = R$,

(3.iii) $\mathrm{Sup}(\alpha) \not\subseteq p$ *for all* $p \in \mathrm{Spec}(R)$,

(3.iv) $\mathrm{Sup}(\alpha) \not\subseteq p$ *for all* $p \in \mathrm{Dp}_1(R)$.

Proof (1) Take $a \in I_{[\alpha]}$. Then $\mathcal{R}(I_{[\alpha]})[1/a] = \mathcal{R}(I_{[\alpha]}[1/a]) = R[1/a]$. So $a \in \mathrm{Sup}(\alpha)$ and hence $I_{[\alpha]} \subseteq \mathrm{Sup}(\alpha)$. In the same manners as in the proof of Theorem 5(1), we can show that $I_{[\alpha]}$ is a radical ideal of R.
(2) By using Theorem 2.3.10 instead of Theorem 3.1.18, our conclusion is obtained in the same manners as in Theorem 3.1.34.
(3) The equivalences (3.i) $\Leftrightarrow$ (3.ii) $\Leftrightarrow$ (3.iii) follow from the same argument as in Theorem 3.1.34, and the equivalence (3.i) $\Leftrightarrow$ (3.iv) follows from Theorem 2.2.8. ∎

From Theorems 3.1.34 and 3.1.35, we have the following result.

Theorem 3.1.36 *Let $\alpha \in L$. Then*

$$\mathrm{Ass}_R(R/\mathrm{Ant}(\alpha)) \subseteq \mathrm{Dp}_1(R)$$

and

$$\mathrm{Ass}_R(R/\mathrm{Sup}(\alpha)) \subseteq \mathrm{Dp}_1(R)$$

Remark 3.1.37 $\mathrm{Sup}(\alpha) \subseteq \mathrm{Ant}(\alpha)$. In fact, if α is super-primitive over R then α is anti-integral over R by Theorem 2.2.8. ∎

Put $I_n := \{a \in R | a \in R$ is the leading coefficient of some polynomial $f(X) \in \mathrm{Ker}(\pi)$ of degree $n\}$. Note that $\mathrm{Ker}(\pi)$ contains no polynomial of degree less than $d = [K(\alpha) : K]$. If $f(X) \in \mathrm{Ker}(\pi)$, then $Xf(X) \in \mathrm{Ker}(\pi)$ and $\deg(Xf(X)) = \deg(f(X)) + 1$. Hence we have an ascending chain:

$$I_d \subseteq I_{d+1} \subseteq \cdots \subseteq I_n \subseteq \cdots$$

Since R is a Noetherian domain, the above chain stops, that is, $I_n = I_{n+1} = \cdots$ for some large n. Put $I_\infty := I_n$.

Proposition 3.1.38 *Under the above notation,*

(1) $I_d = I_{[\alpha]}$;

(2) *for $p \in \mathrm{Spec}(R)$, α is anti-integral over R_p if and only if $p \not\supseteq I_d :_R I_\infty$.*

Proof (1) Take $a \in I_{[\alpha]}$. Then there exists a polynomial $f(X) = aX^d + a_1X^{d-1} + \cdots + a_d \in \mathrm{Ker}(\pi)$. Since $F(\alpha) = 0$ and $\deg(f(X)) = d$, we have $f(X) = a\varphi_\alpha(X) \in R[X]$. Thus $a \in I_{[\alpha]}$. Conversely, let $a\varphi_\alpha(X) \in R[X]$. Then $a \in I_d$ and hence $I_{[\alpha]} \subseteq I_d$. Therefore we have $I_d = I_{[\alpha]}$.
(2) ($\Rightarrow$): Assume that $a \in L$ is anti-integral over R_p. Then the kernel of $\pi_p := \pi \otimes_R R_p : R_p[X] \to R_p[\alpha]$ is generated by some polynomials of degree $d := [K(\alpha)\ K]$. Hence $(I_{[\alpha]})_p = (I\infty)_p$. Thus $p \not\supseteq I_d :_R I_\infty$.
($\Leftarrow$): Assume that $I_d :_R I_\infty \not\subseteq p$ with $p \in \mathrm{Spec}(R)$. Since $(I_d)_p = (I_\infty)_p$, $\mathrm{Ker}(\pi_p)$ is generated by some polynomial of degree d. So α is anti-integral over R_p. [Indeed, if $(f_1(X), \ldots, f_n(X))R_p[X] = \mathrm{Ker}(\pi_p)$ with $\deg(f_i(X)) = d$ for all i, then $f_i(X) = a_i\varphi_i(X)$ with $a_i \in R$, where $\varphi_i(X)$ denotes the monic polynomial in $K[X]$. Since $\deg(\varphi_i(X)) = d$ and $\varphi_i(\alpha) = 0$, $\varphi_1(X) = \cdots = \varphi_n(X)$. Thus $\mathrm{Ker}(\pi_p) = (a_1, \ldots, a_n)\varphi_1(X)R_p[X]$. So we have $(a_1, \ldots, a_n)R_p = I_{[\alpha]}R_p$.] ∎

Combining Theorem 3.1.34 and Proposition 3.1.38, we have the following result.

Corollary 3.1.39

$$\mathrm{Ant}(\alpha) = \sqrt{I_d :_R I_\infty}.$$

3.2 The Integral Closedness of the Ring $R[\alpha] \cap R[\alpha^{-1}]$ (I)

Consider the following statements:

(I) $\{a \in R | a\alpha \text{ is integral over } R\} = I_{[\alpha]}$;

(II) $R\langle\alpha\rangle$ is the integral closure of R in $R[\alpha]$;

(III) $\{a \in R | a\alpha \in R\langle\alpha\rangle\} = I_{[\alpha]}$;

(IV) α is an anti-integral element of degree d over R.

The following implication is shown in Theorem 3.1.6:

$$\text{(III)} \Longleftrightarrow \text{(IV)}$$

First we show the following implications:

$$\text{(II)} + \text{(I)} \Longleftrightarrow \text{(II)} + \text{(IV)}$$

$$\text{(I)} \Longrightarrow \text{(IV)}$$

Proposition 3.2.1 *If $\{a \in R \mid a\alpha \text{ is integral over } R\} = I_{[\alpha]}$ holds, then α is an anti-integral element of degree d over R.*

Proof Assume that $a\alpha \in R\langle\alpha\rangle$, where $a \in R$. Then $a\alpha$ is integral over R by Proposition 3.1.8 (*cf.* [K, p.12 Exercise 4]). Therefore $a \in I_{[\alpha]}$. Hence $\{a \in R \mid a\alpha \in R\langle\alpha\rangle\} \subseteq I_{[\alpha]}$. Next, take an element $a \in I_{[\alpha]}$. By the assumption, $a\alpha$ is integral. So $a\alpha \in R\langle\alpha\rangle$. Hence $\{a \in R \mid a\alpha \in R\langle\alpha\rangle\} = I_{[\alpha]}$. By Corollary 3.1.7 (1) $\Leftrightarrow$ (3), α is an anti-integral element of degree d over R. $\square$

Example 3.2.2 *Let a be a nonzero and nonunit element of R and take an integer $d \geq 2$. Assume that $\varphi_\alpha(X) = X^d + (1/a)X^{d-1} + \cdots + (1/a)^d$ is the minimal polynomial of an algebraic element α over K. Then $a\alpha$ is integral over R. It follows that $(a\alpha)^d + (a\alpha)^{d-1} + \cdots + 1 = 0$, but $a \notin I_{[\alpha]} = a^d R$.*

Remark 3.2.3 In the above example, $R\langle\alpha\rangle$ is not necessarily the integral closure of R in $R[\alpha]$ and $J_{[\alpha]} := I_{[\alpha]}\mathsf{c}(\varphi_\alpha(X)) = R$, where $\mathsf{c}(\varphi_\alpha(X))$ denotes the R-submodule of K generated by the coefficients of $\varphi_\alpha(X)$. Indeed, since a

super-primitive element is an anti-integral element of degree d over R (Theorem 2.2.8, α is an anti-integral element of degree d over R. Suppose that $R\langle\alpha\rangle$ is the integral closure $\overline{R}$ of R in $R[\alpha]$. Since $a\alpha$ is integral over R, it is obvious that $a\alpha \in R\langle\alpha\rangle$. By Theorem 3.1.6, it follows that $a \in I_{[\alpha]}$. This contradicts to the fact that $a \notin I_{[\alpha]} = a^d R$. And so $R\langle\alpha\rangle \subsetneq \overline{R}$. This implies that $a\alpha \in \overline{R}$ and $a\alpha \notin R\langle\alpha\rangle$. ∎

Remark 3.2.4 Assume that $R\langle\alpha\rangle$ is the integral closure of R in $R[\alpha]$. Then $\{a \in R \mid a\alpha \text{ is integral over } R\} = I_{[\alpha]}$ if and only if α is an anti-integral element of degree d over R. ∎

Our second objective of this section is to prove the implication (I) $\Longleftrightarrow$ (II) + (IV) and to show that the implications (II) $\Longrightarrow$ (I), (II) $\Longrightarrow$ (IV), and (IV) $\Longrightarrow$ (I) do not hold by examples.

Definition 3.2.5 Let

$$N_{[\alpha]} := \{a \in R | a\alpha \text{ is integral over } R\}$$

Remark 3.2.6

(i) It is easy to see that $N_{[\alpha]}$ is an ideal of R.

(ii) $N_{[\alpha]} \supseteq I_{[\alpha]}$. Indeed, take $a \in I_{[\alpha]}$. Then putting $a_i = a\eta_i \in R$ $(1 \leq i \leq d)$, we have $a\alpha^d + a_1\alpha^{d-1} + \cdots + a_d = 0$ and hence $(a\alpha)^d + a_1(a\alpha)^{d-1} + \cdots + a_d a^{d-1} = 0$. Thus we have $a\alpha$ is integral over R, which means that $a \in N_{[\alpha]}$. ∎

The implication (IV) $\Longrightarrow$ (I) does not necessarily hold as will be shown in the following example.

Example 3.2.7 Let k be a field and let u, v denote two indeterminates. Put $R := k[u^2, u^3, v^2, v^3] \subseteq k[u, v]$ and $\alpha := v^2/u^2$. Then α is anti-integral over R, i.e., $R\langle\alpha\rangle = R$. Indeed, take $f \in R\langle\alpha\rangle$. Then $f = g(u^2, u^3, v^2, v^3)/u^{2\ell} = h(u^2, u^3, v^2, v^3)/v^{2m}$ for some polynomials $g(W, X, Y, Z)$, $h(W, X, Y, Z) \in k[W, X, Y, Z]$. In this case, we have $u^{2\ell}(h(u^2, u^3, v^2, v^3)/v^{2m}) = g(u^2, u^3, v^2, v^3) \in R$. So

$$h(u^2, u^3, v^2, v^3)/v^{2m} = h_0(u^2, u^3) + v^2 h_2(u^2, u^3, v^2, v^3)$$

where $h_0(W, X)$, $h_2(W, X, Y, Z) \in k[W, X, Y, Z]$. The right hand side of this equality does not contain a linear term in v. By the symmetric argument,

$$g(u^2, u^3, v^2, v^3)/u^{2\ell} = g_0(v^2, v^3) + u^2 g_2(u^2, u^3, v^2, v^3)$$

where $g_0(Y, Z)$, $g_2(W, X, Y, Z) \in k[W, X, Y, Z]$, has no linear term in u. Thus $f \in R$. Moreover $u^3\alpha = uv^2 \in k[u, v] \setminus R$ and the element uv^2 is integral over R, which means that $u^3 \in N_{[\alpha]}$ but $u^3 \notin I_{[\alpha]}$.

Proposition 3.2.8 *If $N_{[\alpha]} = I_{[\alpha]}$, then $R\langle\alpha\rangle$ is the integral closure of R in $R[\alpha]$.*

Proof Since $R\langle\alpha\rangle$ is integral over R, it is enough to prove the following: If $\beta \in R[\alpha]$ is integral over R, then $\beta \in R\langle\alpha\rangle$. Let $\beta = a_n\alpha^n + \cdots + a_1\alpha + a_0 \in R[\alpha]$ be integral over R. Then β has an integral dependence: $\beta^m + b_1\beta^{m-1} + \cdots + b_m = 0$, $(b_i \in R)$. Hence $(a_n\alpha^n + \cdots + a_1\alpha + a_0)^m + b_1(a_n\alpha^n + \cdots + a_1\alpha + a_0)^{m-1} + \cdots + b_m = 0$. Multiplying $a_n^{(n-1)m}$ and expanding it, we have $(a_n\alpha)^{nm} + c_1(a_n\alpha)^{nm-1} + \cdots + c_{nm} = 0$, so that $a_n\alpha$ is integral over R. By the assumption, we have $a_n \in N_{[\alpha]} = I_{[\alpha]}$. When $n \geq d$, $a_n\varphi_\alpha(\alpha) = 0$ holds. So we may assume that $n < d$. By induction on n, we shall show that $\beta \in R\langle\alpha\rangle$. The case $n = 0$ is trivial. Assume that $n > 0$. Put $e_0 = a_n$ and $e_i = a_n\eta_i$, where $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ is the minimal polynomial of α over K. Then $\gamma := e_0\alpha^n + \cdots + e_{d-n} = -(e_{d-n+1}\alpha^{-1} + \cdots + e_d(\alpha^{-1})^{d-n}) \in R\langle\alpha\rangle$. Now $\beta - \gamma$ is integral over R, and so $\beta - \gamma \in R\langle\alpha\rangle$ by induction hypothesis. Thus $\beta \in R\langle\alpha\rangle$. ∎

Proposition 3.2.9 *Assume that α is an anti-integral element over R. Then $\sqrt{N_{[\alpha]}} = \sqrt{I_{[\alpha]}}$.*

Proof By Remark 3.2.6(ii), we have $I_{[\alpha]} \subseteq N_{[\alpha]}$. So $\sqrt{I_{[\alpha]}} \subseteq \sqrt{N_{[\alpha]}}$. Take $a \in N_{[\alpha]}$. Since $a\alpha$ is integral over R, we have $(a\alpha)^n + a_1(a\alpha)^{n-1} + \cdots + a_n = 0$. Since α is anti-integral over R, so is α^{-1} by Theorem 2.4.5. Then from Lemma 3.1.10, it follows that $a^n \in I_{[\alpha]}$. ∎

As a corollary, we have the following result.

Corollary 3.2.10 *Assume that α is an anti-integral element over R. If $I_{[\alpha]}$ is a radical ideal of R, then $N_{[\alpha]} = I_{[\alpha]}$.*

Proof By Proposition 3.2.9, we have $N_{[\alpha]} \subseteq \sqrt{N_{[\alpha]}} = \sqrt{I_{[\alpha]}} = I_{[\alpha]}$. Hence $N_{[\alpha]} = I_{[\alpha]}$ by Remark 3.2.6(ii) because $N_{[\alpha]}$ is a radical ideal of R. ∎

The following example shows that even if $R\langle\alpha\rangle$ is the integral closure of R in $R[\alpha]$, α is not necessarily anti-integral and $N_{[\alpha]} = I_{[\alpha]}$ does not always hold.

Example 3.2.11 Let k be a field and let t be an indeterminate. Put $R := k[t^2, t^3]$ and $\alpha := t$. Then $R[\alpha] = R[t] = k[t]$ and $R[\alpha^{-1}] = R[1/t] = k[t, 1/t]$. Thus $R\langle\alpha\rangle = k[t]$ is the integral closure of R in $R[\alpha] = k[t]$. But α is not anti-integral because $R\langle\alpha\rangle \neq R$. Moreover $N_{[\alpha]} = R$ and $I_{[\alpha]} = (t^2, t^3)R$.

Theorem 3.2.12 *The following statements are equivalent:*

(1) $N_{[\alpha]} = I_{[\alpha]}$:

(2) *α is anti-integral over R and $R\langle\alpha\rangle$ is the integral closure of R in $R[\alpha]$.*

Proof (1) $\Rightarrow$ (2) follows from Propositions 3.2.1 and 3.2.8.
(2) $\Rightarrow$ (1) follows from the equivalence: (II) + (I) $\Rightarrow$ (II) + (IV). $\square$

As a corollary, we have the following result.

Corollary 3.2.13 *Let α be an element in K and assume that α is anti-integral over R. If the equality $R :_R \alpha = \overline{R} :_R \alpha$ holds, then R is integrally closed in $R[\alpha]$.*

Proof In the birational case, we have $R :_R \alpha = I_\alpha = I_{[\alpha]}$ and $\overline{R} :_R \alpha = N_{[\alpha]}$. Hence the assumption implies that $I_{[\alpha]} = N_{[\alpha]}$. Thus $R = R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$ by Theorem 3.2.12. $\square$

Let $A \subseteq B$ be two integral domains. We denote by $\mathrm{c}(B/A)$ the ideal $\{a \in A | aB \subseteq A\}$ of A. We also say that *the dimension formula holds* between A and B if $\mathrm{ht}(P) = \mathrm{ht}(p) + \mathrm{Tr.deg}_A B - \mathrm{Tr.deg}_{k(p)} k(P)$ for every $P \in \mathrm{Spec}(B)$, where $p = P \cap A$ (*cf.* [M2, p.119]). It is clear that if B is a finite A-module, then the dimension formula (if it holds) implies that $\mathrm{ht}(P) = \mathrm{ht}(P \cap A)$ for each $P \in \mathrm{Spec}(B)$.

Recall for $\alpha \in K$, I_α denotes the ideal $\{a \in R | a\alpha \in R\}$ of R.

Proposition 3.2.14 *Assume the following conditions:*

(i) *the integral closure $\overline{R}$ of R in K is a finite R-module;*

(ii) *the dimension formula holds between R and $\overline{R}$;*

(iv) *$\alpha_1, \ldots, \alpha_n \in K$ are super-primitive over R.*

Then R is integrally closed in $R[\alpha_1, \ldots, \alpha_n]$ if and only if $\mathrm{grade}(I_{\alpha_1} \cap \cdots \cap I_{\alpha_n} + \mathcal{C}(\overline{R}/R)) > 1$.

Proof ($\Rightarrow$): Suppose that there exists $p \in \mathrm{Dp}_1(R)$ such that $I_{\alpha_1} \cap \cdots \cap I_{\alpha_n} + \mathcal{C}(\overline{R}/R) \subseteq p$. Then $I_{\alpha_1} \cap \cdots \cap I_{\alpha_n} \subseteq p$ and hence $I_{\alpha_i} \subseteq p$ for some i. Since α_i is

a super-primitive element over R, I_{α_i} is divisorially flat. So p is a prime divisor of I_{α_i}. Thus there exists $a \in R$ such that $I_{a\alpha_i} = p$. Suppose that $\mathrm{ht}(p) > 1$. Then $a\alpha_i \in \overline{R} \setminus R$ by the dimension formula. [Indeed, $P \in \mathrm{Ht}_1(\overline{R})$ satisfies $P \cap R \in \mathrm{Ht}_1(R)$ by the dimension formula. The inclusion $a\alpha_i p \subseteq R \subseteq \overline{R}$ implies $a\alpha_i \in R_{P\cap R} \subseteq \overline{R}_P$ and hence $a\alpha_i \in \bigcap_{P\in \mathrm{Ht}_1(\overline{R})} \overline{R}_P = \overline{R}$ because $\overline{R}$ is a Krull domain.] Hence $a\alpha_i \in R[\alpha_i] \subseteq R[\alpha_1, \ldots, \alpha_n]$, which contradicts the assumption that R is integrally closed in $R[\alpha_1, \ldots, \alpha_n]$. Next suppose that $\mathrm{ht}(p) = 1$. Since $\mathcal{C}(\overline{R}/R) \subseteq p$, p is a conductor ideal. So $(a\alpha_i)p \subseteq p$ and hence $a\alpha_i \in \overline{R}$, a contradiction.

($\Leftarrow$): Suppose that there exists $\beta \in R[\alpha_1, \ldots, \alpha_n]$ with $\beta \in \overline{R} \setminus R$. Since β is expressed as $\sum b_{i_1\cdots i_n}\alpha_1^{i_1}\cdots\alpha_n^{i_n}$, any element $a \in I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$ satisfies $a^N\beta \in R$ for a sufficiently large integer N. Thus $I_{\alpha_1} \cap \cdots \cap I_{\alpha_n} \subseteq \sqrt{I_\beta}$. Take a prime divisor p of I_β. Then $\beta \in R$ implies $\mathcal{C}(\overline{R}/R) = R :_R \overline{R} \subseteq R :_R \beta = I_\beta \subseteq p$. Thus $\mathcal{C}(\overline{R}/R) \subseteq p$ and hence $I_{\alpha_1} \cap \cdots \cap I_{\alpha_n} + \mathcal{C}(\overline{R}/R) \subseteq p$. Since I_β is divisorial ideal, $\mathrm{depth}(R_p) = 1$, which contradicts the assumption $\mathrm{grade}(I_{\alpha_1} \cap \cdots \cap I_{\alpha_n} + \mathcal{C}(\overline{R}/R)) > 1$. ∎

Theorem 3.2.15 *Assume that $\alpha \in K$ is a super-primitive element over R and that the dimension formula holds between R and $\overline{R}$. Then the following statements are equivalent:*

(1) *R is integrally closed in $R[\alpha]$;*

(2) *putting $N_\alpha := \{x \in R | x\alpha \in \overline{R}\}$, $I_\alpha = N_\alpha$ holds;*

(3) *none of the prime divisors p of I_α contains $\mathcal{C}(\overline{R}/R)$;*

(4) $\mathrm{grade}(I_\alpha + \mathcal{C}(\overline{R}/R)) > 1$.

Proof Since α is super-primitive over R, α is anti-integral over R by Theorem 2.2.8. Hence $R\langle\alpha\rangle = R$. Thus (1) $\Leftrightarrow$ (2) follows from Theorem 3.2.12. (1) $\Leftrightarrow$ (4) follows from Proposition 3.2.14.

(1) $\Rightarrow$ (3): Let p be a prime divisor of I. Then there exists $a \in R$ such that $I_{a\alpha} = p$. Assume that $\mathcal{C}(\overline{R}/R) \subseteq p$. Then $\mathrm{ht}(p) = 1$. [Indeed, suppose that $\mathrm{ht}(p) > 1$. Then the dimension formula and $\mathrm{depth}(R_p) = 1$ imply $a\alpha \in \overline{R}$ (*cf.* the proof of Proposition 3.2.14). Since $a\alpha \notin R$ and $a\alpha \in R[\alpha]$, this is a contradiction.] Now since $\mathcal{C}(\overline{R}/R) \subseteq p$, p is a minimal prime divisor of $\mathcal{C}(\overline{R}/R)$. Hence $(a\alpha)p \subseteq p$. [In fact, suppose that $(a\alpha)p \not\subseteq p$. Then $(a\alpha)pR_p = R_p$ and hence $1 = (a\alpha)\cdot y$ for some $y \in R$. So we have $pR_p = yR_p$, which means that R_p is a DVR and hence $p \not\supseteq \mathcal{C}(\overline{R}/R)$, a contradiction.] Thus $a\alpha \in \overline{R}$, $a\alpha \notin R$ and $a\alpha \in R[\alpha]$, which is a contradiction. Therefore $p \not\supseteq \mathcal{C}(\overline{R}/R)$.

(3) $\Rightarrow$ (1): Suppose that there exists $\beta \in R[\alpha]$ such that $\beta \in \overline{R} \setminus R$. Then we can write $\beta = a_0 + a_1\alpha + \cdots + a_n\alpha^n$ with $a_i \in R$. So for any $a \in I_\alpha$, we have $a^n\beta \in R$. Thus we have $I_\alpha \subseteq \sqrt{I_\beta}$. Let p be a prime divisor of $\sqrt{I_\beta}$.

Since $\beta \in \overline{R}$, $p \in \mathrm{Ass}_R(\overline{R}/R)$. Hence $p \supseteq \mathcal{C}(\overline{R}/R)$. Thus $p \supseteq I_\alpha + \mathcal{C}(\overline{R}/R)$, which is a contradiction. Therefore R is integrally closed in $R[\alpha]$. $\square$

Theorem 3.2.16 *Let L be an algebraic field extension of K, $\alpha \in L$, and $\widetilde{R}$ denote the integral closure of R in $K(\alpha)$. Assume that the dimension formula holds between R and $\widetilde{R}$, and that $\widetilde{R}$ is a finite R-module. Then the following statements are equivalent:*

(1) *$R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$;*

(2) *$I_{[\alpha]} = N_{[\alpha]}$;*

(3) $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\widetilde{R}/R\langle\alpha\rangle) \cap R) > 1$;

(4) *none of the prime divisors p of $I_{[\alpha]}$ contains $\mathcal{C}(\widetilde{R}/R\langle\alpha\rangle) \cap R$.*

Proof (1) $\Rightarrow$ (3): Take $p \in \mathrm{Dp}_1(R)$ such that $I_{[\alpha]} + \mathcal{C}(\widetilde{R}/R\langle\alpha\rangle) \cap R \subseteq p$. Since α is super-primitive over R, $I_{[\alpha]}$ is divisorially flat. So p is a prime divisor of $I_{[\alpha]}$. It follows that $I_{[\alpha]} = N_{[\alpha]} = R\langle\alpha\rangle :_R \alpha$ from Theorem 3.2.12 and Definition 3.2.5. Hence there exists an element $a \in R$ such that $p = R\langle\alpha\rangle :_R a\alpha$. Suppose now that $\mathrm{ht}(p) > 1$. Take $P \in \mathrm{Spec}(\widetilde{R})$ with $\mathrm{ht}(P) = 1$. Then $\mathrm{ht}(P \cap R) = 1$ by the dimension formula. Thus $P \cap R \neq p$, which implies that $(a\alpha)p \subseteq R\langle\alpha\rangle \subseteq \widetilde{R}$ and hence $a\alpha \in \widetilde{R}_P$. So we have $a\alpha \in \bigcap \widetilde{R}_P = \widetilde{R}$. Thus we have $a\alpha \in R[\alpha]$ and $a\alpha \notin R\langle\alpha\rangle$. This contradicts the assumption that $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$. Therefore $\mathrm{ht}(p) = 1$. In this case, $\mathcal{C}(\widetilde{R}/R\langle\alpha\rangle) \cap R \subseteq p$ holds. Thus there exists a prime ideal P of $R\langle\alpha\rangle$ such that $\mathcal{C}(\widetilde{R}/R\langle\alpha\rangle) \subseteq P$ with $P \cap R = p$. Since $R\langle\alpha\rangle$ is integral over R, we have $\mathrm{ht}(P) = 1$. So P is a minimal prime divisor of $\mathcal{C}(\widetilde{R}/R\langle\alpha\rangle)$. Hence $a\alpha P \subseteq P$ (*cf.* the proof of Theorem 3.2.15). Thus $a\alpha \in R[\alpha]$ and $a\alpha \notin R\langle\alpha\rangle$. This contradicts the assumption that $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$. Therefore we conclude that $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\widetilde{R}/R\langle\alpha\rangle) \cap R) > 1$.
(3) $\Rightarrow$ (1): Take $a \in I_{[\alpha]}$. Then putting $a\eta_i = a_i$, $a\alpha^d + a_1\alpha^{d-1} + \cdots + a_d = 0$. So we have $a\alpha = -(a_1 + a_2\alpha^{-1} + \cdots + a_d(\alpha^{-1})^{d-1}) \in R\langle\alpha\rangle$. Conversely, if $a \in R$ and $a\alpha \in R\langle\alpha\rangle$, then $a\alpha = (b_0 + b_1\alpha^{-1} + \cdots + b_n(\alpha^{-1})^n)$. Thus $a\alpha^{n+1} + b_0\alpha^n + \cdots + b_n = 0$. Since α is anti-integral over R, so is α^{-1}. Hence $a \in I_{[\alpha]}$. Therefore $R\langle\alpha\rangle :_R \alpha = I_{[\alpha]}$. Take $\beta \in R[\alpha]$ with $\beta \in \widetilde{R} \setminus R\langle\alpha\rangle$. Then $\beta = \sum b_i\alpha^i$ for $b_i \in R$. Since $R\langle\alpha\rangle :_R \alpha = I_{[\alpha]}$, there exists a sufficiently large integer N such that any $b \in I_{[\alpha]}$ satisfies $b^N\beta \in R\langle\alpha\rangle$. Let p be a prime divisor of $R\langle\alpha\rangle :_R \beta$. Then $I_{[\alpha]} \subseteq \sqrt{R\langle\alpha\rangle :_R \beta} \subseteq p$ and $\mathcal{C}(\widetilde{R}/R\langle\alpha\rangle) \cap R = R(\alpha) :_R \widetilde{R} \subseteq R\langle\alpha\rangle :_R \beta \subseteq p$. So $I_{[\alpha]} + \mathcal{C}(\widetilde{R}/R\langle\alpha\rangle) \cap R \subseteq p$. We have only to show that $\mathrm{depth}(R_p) = 1$. Let $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ be the minimal polynomial of α over K. Put $\zeta_i = \alpha^{d-i} + \eta_1\alpha^{d-i-1} + \cdots + \eta_{d-i}$. Then $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ as R-modules (*cf.* Proposition 3.2.19). Since $L = R\langle\alpha\rangle \otimes_R K$, $L = K + K\zeta_1 + \cdots + K\zeta_{d-1}$. Write $\beta = \lambda_0 + \lambda_1\zeta_1 + \cdots + \lambda_{d-1}\zeta_{d-1}$. Then $x\beta \in R\langle\alpha\rangle$ if and only if $x\lambda_0 \in R$, $x\lambda_1 \in I_{[\alpha]}, \ldots, x\lambda_{d-1} \in I_{[\alpha]}$. Thus $R\langle\alpha\rangle :_R \beta$ is a

divisorial ideal $(R : \lambda_0) \cap (I_{[\alpha]} : \lambda_1) \cap \cdots \cap (I_{[\alpha]} : \lambda_{d-1})$. Since p is a prime divisor of this ideal, we have $\mathrm{depth}(R_p) = 1$. Thus $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\widetilde{R}/R\langle\alpha\rangle) \cap R) = 1$, a contradiction. Therefore $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$.
(1) $\Leftrightarrow$ (2) follows from Theorem 3.2.12, and (3) $\Leftrightarrow$ (4) is obtained by the similar way to the proof (3) $\Leftrightarrow$ (4) of Theorem 3.2.15. $\square$

Proposition 3.2.17 *Let k be a field contained in R and assume that α is an anti-integral element over R of degree d. If $I_{[\alpha]} = N_{[\alpha]}$, then $R\langle\alpha\rangle = R[\alpha + \lambda] \cap R[1/(\alpha + \lambda)]$ for each $\lambda \in k$.*

Proof Note that $N_{[\alpha]} = \{a \in R | a\alpha$ is integral over $R\}$ and that $a\alpha$ is integral over R if and only if $a(\alpha + \lambda)$ is integral over R. Hence $N_{[\alpha]} = N_{[\alpha+\lambda]}$. It is easy to see that $I_{[\alpha]} = I_{[\alpha+\lambda]}$. Thus we have $I_{[\alpha]} = N_{[\alpha]}$ and $I_{[\alpha+\lambda]} = N_{[\alpha+\lambda]}$. So both $R\langle\alpha\rangle$ and $R[\alpha + \lambda] \cap R[1/(\alpha + \lambda)]$ are the integral closure of R in $R[\alpha] = R[\alpha + \lambda]$, which means that $R\langle\alpha\rangle = R[\alpha + \lambda] \cap R[1/(\alpha + \lambda)]$. $\square$

Let J be an ideal of R and let R' be a ring containing R. An element $b \in R'$ is said to be *integral* over J if there exist elements $a_1, \ldots, a_n$ such that $a_i \in J^i$ for each i and such that $b^n + a_1 b^{n-1} + \cdots + a_n = 0$. When $R' = R$, the set of the elements integral over J is an ideal of R, which is called the *integral closure* of J and is denoted by $\overline{J}$.

Proposition 3.2.18 *Assume that α is anti-integral over R of degree d. If $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$, then $I_{[\alpha]} = \overline{I_{[\alpha]}}$.*

Proof Take $x \in \overline{I_{[\alpha]}}$. Then $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ with $a_i \in (I_{[\alpha]})^i$. Since $I_{[\alpha]}\alpha = N_\alpha \alpha$ is integral over R by Theorem 3.2.12 and is contained in $R[\alpha]$, $a_i\alpha^i \in R[\alpha]$ is integral over R. Thus $(x\alpha)^n + a_1\alpha(x\alpha)^{n-1} + (a_2\alpha^2)(x\alpha)^{n-2} + \cdots + a_n\alpha^n = 0$, which means that $x\alpha$ belongs to $R[\alpha]$ and is integral over R. Hence $x \in N_{[\alpha]} = I_{[\alpha]}$ by Theorem 3.2.12. Therefore we have $\overline{I_{[\alpha]}} = I_{[\alpha]}$. $\square$

3.3 The Integral Closedness of the Ring $R[\alpha] \cap R[\alpha^{-1}]$ (II)

We start with the following lemma.

Lemma 3.3.1 *Every nonzero principal ideal of R is integrally closed if and only if R is integrally closed in K.*

Proof Assume that $a/b \in K$ $(a, b \in R)$ is integral over R. Then there exists an integral dependence over R:

$$(a/b)^r + c_1(a/b)^{r-1} + \cdots + c_r = 0$$

We have

$$a^r + bc_1a^{r-1} + \cdots + b^rc_r = 0$$

Since the principal ideal bR is integrally closed, it follows that $a \in bR$, and hence $a/b \in R$. Conversely consider a principal ideal bR and an integral dependence over bR:

$$a^r + c_1a^{r-1} + \cdots + c_r = 0$$

Then we can put $c_i = h_ib^i$ for some $h_i \in R$ by definition, we have:

$$(a/b)^r + h_1(a/b)^{r-1} + \cdots + h_r = 0$$

Since R is integrally closed in K, it follows that $a/b \in R$, that is, $a \in bR$. So bR is integrally closed. ∎

The converse implication of Proposition 3.2.18 is not always valid as will be seen in the following example.

Example 3.3.2 Let R be a polynomial ring $k[a]$ over a field k, which is a normal domain. Let $\varphi_\alpha(X) = X^2 + (1/a)X + (1/a^2)$. Then $I_{[\alpha]} = a^2R$. By [KY, Theorem 1], $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 = R \oplus I_{[\alpha]}(\alpha + (1/a)) = R \oplus (a^2\alpha + a)R$, as R-modules. Note that $a\alpha \in R[\alpha]$ and $a\alpha \notin R\langle\alpha\rangle$. Since $a^2\alpha^2 + a\alpha + 1 = 0$, we have $(a\alpha)^2 = a^2\alpha^2 = -(a\alpha + 1)$ and $(a\alpha)^3 = a\alpha(a\alpha)^2 = -a\alpha(a\alpha + 1) = -(a\alpha)^2 - a\alpha = (a\alpha + 1) - a\alpha = 1$. Hence $a\alpha$ is integral over $R\langle\alpha\rangle$, which means that $R\langle\alpha\rangle$ is not integrally closed in $R[\alpha]$. Since R is integrally closed in K, the ideal $I_{[\alpha]} = a^2R$ is integrally closed by Lemma 3.3.1.

Lemma 3.3.3 *Let I be an ideal of R. Then I is integrally closed if and only if I_p is integrally closed for each $p \in \mathrm{Ass}_R(R/I)$.*

Proof $(\Rightarrow)$: Take $p \in \mathrm{Ass}_R(R/I)$, let $x \in R_p$ be integral over I_p. Then we have an integral dependence: $x^\ell + a_1x^{\ell-1} + \cdots + a_\ell = 0$ with $a_i \in (I_p)^i$ $(1 \le i \le \ell)$. We can find an element $s \in R \setminus p$ such that $sa_i \in I^i$ $(1 \le i \le \ell)$ and $sx \in R$. Hence we obtain $(sx)^\ell + (sa_1)(sx)^{\ell-1} + \cdots + s^\ell a_\ell = 0$ with $sa_i \in I^i$ $(1 \le i \le \ell)$. Since I is integrally closed, we have $sx \in I$. Thus $x \in I_p$.

$(\Leftarrow)$: Take $x \in R$ which is integral over I. Then there is an integral dependence: $x^\ell + a_1x^{\ell-1} + \cdots + a_\ell = 0$ with $a_i \in I^i$ $(1 \le i \le \ell)$. So we have

$a \in I^i \subseteq (I_p)^i$ for each $p \in \mathrm{Ass}_R(R/I)$. Since I_p is integrally closed, we have $x \in I_p$. Hence $x \in \bigcap_{p \in \mathrm{Ass}_R(R/I)} I_p = I$. Thus I is integrally closed. $\square$

Let $\mathcal{C}(\overline{R}/R)$ denote the conductor between R and $\overline{R}$, that is, $\mathcal{C}(\overline{R}/R) = \{a \in R | a\overline{R} \subseteq R\}$.

Theorem 3.3.4 *Assume that $\overline{R}$ is a finitely generated R-module and that the dimension formula holds between R and $\overline{R}$. Assume that α is a super-primitive element of degree d over R. Then the following statements are equivalent to each other:*

(i) *the ideal $I_{[\alpha]}$ is integrally closed;*

(ii) $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$.

Proof (ii) $\Rightarrow$ (i): Let p be a prime divisor of $I_{[\alpha]}$. Since $I_{[\alpha]}$ is a divisorial ideal, we have $p \in \mathrm{Dp}_1(R)$. Since α is super-primitive over R, we have $(I_{[\alpha]})_p = aR_p$ for some $a \in I_{[\alpha]}$. Thus the assumption $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$ yields that $\mathcal{C}(\overline{R}/R) \not\subseteq p$. Hence $R_p = \overline{R}_p$, which means that $(I_{[\alpha]})_p$ is integrally closed for each $p \in \mathrm{Ass}_R(R/I_{[\alpha]})$ by Lemma 3.3.1. So we conclude that $I_{[\alpha]}$ is integrally closed by Lemma 3.3.3.
(i) $\Rightarrow$ (ii): Note first that $\mathcal{C}(\overline{R}/R) \neq 0$ because $\overline{R}$ is a finitely generated R-module. Suppose that $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) \leq 1$. Then there exists a prime ideal $p \in \mathrm{Dp}_1(R)$ such that $I_{[\alpha]} + \mathcal{C}(\overline{R}/R) \subseteq p$. We now claim that $p \in \mathrm{Ass}_R(\overline{R}/R)$. [In fact, when $\mathrm{ht}(p) = 1$, the inclusion $\mathcal{C}(\overline{R}/R) \subseteq p$ implies that p is a minimal prime divisor of $\mathrm{Ann}_R(\overline{R}/R)$ and hence that $p \in \mathrm{Ass}_R(\overline{R}/R)$. Assume that $\mathrm{ht}(p) > 1$. Since $\mathrm{depth}(R_p) = 1$, there exists $\beta \in K$ such that $I_\beta = p$. Since $\overline{R}$ is a Krull domain, $\overline{R} = \bigcap_{P \in \mathrm{Ht}_1(\overline{R})} \overline{R}_P$. Take $P \in \mathrm{Ht}_1(\overline{R})$ and put $q := P \cap R$. Then by the dimension formula, we have $\mathrm{ht}(q) = 1$. Since $\mathrm{ht}(p) > 1$, it follows that $q \not\subseteq p = I_\beta$. Thus $\beta \in R_q \subseteq \overline{R}_P$ and so $\beta \in \bigcap \overline{R}_P = \overline{R}$. Hence we have $p \in \mathrm{Ass}_R(\overline{R}/R)$. Therefore we have shown that $p \in \mathrm{Ass}_R(\overline{R}/R)$ in any case.] Since $p \in \mathrm{Ass}_R(\overline{R}/R)$ by the above claim, there exists $\gamma \in \overline{R}$ such that $\gamma \notin R$ and $p = I_\gamma$. Since α is super-primitive and $p \in \mathrm{Dp}_1(R)$, $J_{[\alpha]}R_p = R_p$ and hence $(I_{[\alpha]})_p$ is invertible. So there exists $a \in R$ such that $(I_{[\alpha]})_p = aR_p$. The implications $aR_p = (I_{[\alpha]})_p \subseteq pR_p = I_\gamma R_p$ yield that $a\gamma \in R_p$ and $\gamma \notin R_p$. Since $\gamma \in \overline{R}_p$, there exists an integral dependence:

$$\gamma^n + b_1\gamma^{n-1} + \cdots + b_n = 0$$

with $b_i \in R_p$ for some n. Hence we have $(a\gamma)^n + (ab_1)(a\gamma)^{n-1} + \cdots + a^n b_n = 0$. Since $I_{[\alpha]}$ is integrally closed, so is $(I_{[\alpha]})_p = aR_p$. Thus $a\gamma \in aR_p$ and hence $\gamma \in R_p$, a contradiction. Therefore $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$. $\square$

Corollary 3.3.5 *Assume that $\overline{R}$ is a finitely generated R-module and that the dimension formula holds between R and $\overline{R}$. Assume that α is a super-primitive element of degree d over R. If $I_{[\alpha]}$ is integrally closed, then R_p is a DVR for each $p \in \mathrm{Ass}_R(R/I_{[\alpha]})$.*

Proof Since $I_{[\alpha]}$ is integrally closed, we have $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$ by Theorem 3.3.4. Take $p \in \mathrm{Ass}_R(R/I_{[\alpha]})$. Since $I_{[\alpha]} = \bigcap_{1 \le i \le d} I_{\eta_i}$ is a divisorial ideal, the ideal p is divisorial prime ideal of R. Hence $\mathrm{grade}(p) \le 1$. Thus $\mathcal{C}(\overline{R}/R) \not\subseteq p$, which means $\overline{R}_p = R_p$. So we have $p \in \mathrm{Ht}_1(R)$ and R_p is a Noetherian normal local domain of dimension 1, that is, a DVR. ∎

Recall that $N_{[\alpha]} := \{a \in R | a\alpha \text{ is integral over } R\}$.

Proposition 3.3.6 *Assume that $a\alpha$ is anti-integral over R for all $a \in R$. Then $N_{[\alpha]} = \{a \in R | a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in R\}$.*

Proof Let a be an element in R such that $a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in R$. Put $b_1 = a\eta_1, b_2 = a^2\eta_2, \ldots, b_d = a^d\eta_d$. Then $(a\alpha)^d + b_1(a\alpha)^{d-1} + \cdots + b_d = 0$. Hence $a\alpha$ is integral over R, which means that $a \in N_{[\alpha]}$ by the definition. Conversely, take $a \in N_{[\alpha]}$. Then $a\alpha$ is integral over R. Since $a\alpha$ is anti-integral over R by the assumption, the element $a\alpha$ satisfies a monic polynomial equation of degree d over R by Theorem 2.3.2, that is, $\varphi_{a\alpha}(X) \in R[X]$. Since $\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0$, we have $(a\alpha)^d + a\eta_1(a\alpha)^{d-1} + \cdots + a^d\eta_d = 0$. Since $\deg \varphi_{a\alpha}(X) = d$, comparing the coefficients of $\varphi_{a\alpha}(a\alpha)$ with those of $(a\alpha)^d + a\eta_1(a\alpha)^{d-1} + \cdots + a^d\eta_d$ in $K[\alpha]$, we have $a\eta_1, \ldots, a^d\eta_d \in R$. Hence $a \in \{a \in R | a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in R\}$. ∎

Combining Proposition 3.3.6 with Theorem 3.3.12, we have the following result:

Proposition 3.3.7 *Assume that $a\alpha$ is anti-integral over R for all $a \in R$. $R\langle\alpha\rangle$ is integrally closed if and only if $I_{[\alpha]} = \{a \in R | a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in R\}$.*

Proposition 3.3.8 *Assume that α is super-primitive over R. If $I_{[\alpha]}$ is integrally closed, then $a\alpha$ is anti-integral over R for every nonzero $a \in R$.*

Proof Since α is super-primitive over R, $(J_{[\alpha]})_p = R_p$ for every $p \in \mathrm{Dp}_1(R)$, which means that $(I_{[\alpha]})_p$ is invertible, that is, $(I_{[\alpha]})_p$ is a principal ideal of R_p. So $R_p = \overline{R}_p$ by Lemma 3.3.1 because that $I_{[\alpha]}$ is integrally closed (and hence $(I_{[\alpha]})_p$ is integrally closed). Since $R_p - \overline{R}_p$ for every $p \in \mathrm{Dp}_1(R)$, $a\alpha$

is super-primitive over R_p for all $p \in \mathrm{Dp}_1(R)$ by Theorem 2.2.8. Hence $a\alpha$ is anti-integral over R_p for all $p \in \mathrm{Dp}_1(R)$ by Theorem 2.2.8, which implies that $a\alpha$ is anti-integral over R by Theorem 2.4.3. ∎

Proposition 3.3.9 *Assume that α is a super-primitive element of degree d over R. Then the following statements are equivalent:*

(i) $I_{[\alpha]} = I_{\eta_1}$ *and this ideal is integrally closed;*

(ii) $R\langle\alpha\rangle$ *is integrally closed in* $R[\alpha]$ *and* $I_{\eta_1}^i \subseteq I_{\eta_i}$ *for all* i $(1 \le i \le d)$.

Proof (ii) $\Rightarrow$ (i): Since $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$, $I_{[\alpha]}$ is integrally closed by Proposition 3.2.18 and $N_{[\alpha]} = I_{[\alpha]}$ by Theorem 3.2.12. Take $a \in I_{\eta_1}$. Then $a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in R$ by the assumption. Thus $a\alpha$ satisfies a monic equation of degree d over R, that is, $\varphi_{a\alpha}(a\alpha) = (a\alpha)^d + a\eta_1(a\alpha)^{d-1} + \cdots + a^d\eta_d = 0$. So $a\alpha$ is integral over R, which means that $a \in N_{[\alpha]} = I_{[\alpha]}$. Thus $I_{\eta_i} \subseteq I_{[\alpha]}$. The inclusion $I_{[\alpha]} \subseteq I_{\eta_1}$ is obvious by definition.
(i) $\Rightarrow$ (ii): Since $I_{\eta_1} = I_{[\alpha]} \subseteq I_{\eta_i}$, we have $I_{\eta_1}^i \subseteq I_{\eta_i}$. We must show that $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$. For this end, we have only to prove that $I_{[\alpha]} = N_{[\alpha]}$ by Theorem 3.2.12. Take $a \in N_{[\alpha]}$. Then $a\alpha$ is integral over R. Since $I_{[\alpha]}$ is integrally closed and since α is super-primitive over R, $a\alpha$ is anti-integral over R by Proposition 3.3.8. Thus $a\alpha$ satisfies a monic polynomial equation of degree d over R by Theorem 2.3.2, that is, $\varphi_{a\alpha}(X) \in R[X]$. Since $\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0$ in $K[\alpha]$, we have $(a\alpha)^d + a\eta_1(a\alpha)^{d-1} + \cdots + a^d\eta_d = 0$. So $a^i \in I_{\eta_i}$ $(1 \le i \le d)$. Since $\deg \varphi_{a\alpha}(X) = d$, comparing the coefficients of $\varphi_{a\alpha}(a\alpha)$ with those of $(a\alpha)^d + a\eta_1(a\alpha)^{d-1} + \cdots + a^d\eta_d$, we have $a\eta_1, \ldots, a^d\eta_d \in R$. Hence $a \in I_{\eta_1} = I_{[\alpha]}$. Therefore we obtain $I_{[\alpha]} = N_{[\alpha]}$. ∎

The following result determines the ideal $N_{[\alpha]}$.

Theorem 3.3.10 $N_{[\alpha]} = \{a \in R | a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in \overline{R}\}$.

Proof Take $a \in R$ such that $a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in \overline{R}$. Since $(a\alpha)^d + a\eta_1(a\alpha)^{d-1} + \cdots + a^d\eta_d = 0$, $a\alpha$ is integral over $\overline{R}$ and hence is integral over R. Hence $a \in N_{[\alpha]}$. Conversely, take $a \in N_{[\alpha]}$. Then $a\alpha$ is integral over R by definition and hence is integral over $\overline{R}$. Since $\overline{R}$ is a Krull domain, $a\alpha$ has a monic relation of degree d over R by Gauss' lemma. The desired monic relation must be $(a\alpha)^d + a\eta_1(a\alpha)^{d-1} + \cdots + a^d\eta_d = 0$. Hence $a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in \overline{R}$. ∎

Theorem 3.2.12 and Theorem 3.3.10 give rise to the following corollary.

Corollary 3.3.11 *$R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$ if and only if*

$$I_{[\alpha]} = \{a \in R | a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in \overline{R}\}$$

Example 3.3.12 Even if $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$, the ideal $I_{[\alpha]}$ is not always a radical ideal of R. For instance, let R be a polynomial ring $k[a]$ over a field k, and let $\varphi_\alpha(X) = X^2 + (1/a^2)X + (1/a^2)$. Then $I_{[\alpha]} = a^2R$. Since $(a^2\alpha)^2 + (a^2\alpha) + a^2 = 0$, we have $N_{[\alpha]} = a^2R$. So $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$ by Theorem 3.2.12. But $I_{[\alpha]}$ is not a radical ideal of R.

Let (V, m) be a DVR and let J be a nonzero ideal of V. Then m and J are principal ideals tV and $t^\ell V$ for some $\ell \in \mathbf{N}$, respectively. Let v denote the valuation associated with V. In this case, we write $v(J) := \ell$.

To obtain the next result, we recall the following:

Assume that α is an anti-integral element of degree d over R. Let $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ be the minimal polynomial of α over K. Put $\zeta_i = \alpha^{d-i} + \eta_1\alpha^{d-i-1} + \cdots + \eta_{d-i}$ $(1 \le i \le d-1)$. Then $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ as R-modules (*cf.* Proposition 3.1.19).

Lemma 3.3.13 *Assume that R is a* DVR *with maximal ideal m and that $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$. If $\eta_1 \in R$, then $I_{[\alpha]} = m$, that is, $v(I_{[\alpha]}) = 1$, where $v(\ )$ denotes the valuation associated with R.*

Proof Put $v(I_{[\alpha]}) = e$. Since $I_{[\alpha]} = \bigcap_{i=1}^d I_{\eta_i}$, $I_{[\alpha]} \subseteq I_{\eta_i}$. So $v(\eta_i) + e \ge 0$ and hence $v(\eta_i) \ge -e$ $(1 \le i \le d-1)$.

Suppose that $e \ge 3$. Take $a \in R$ such that $v(a) = e - 1$. Then $a \notin I_{[\alpha]}$. Consider the element $(a\zeta_1)^3$. Then

$$(a\zeta_1)^3 = a^3(-(\eta_1\eta_2 + \eta_3) + (\eta_1^2 - \eta_1)\zeta_1 + 2\eta_1\zeta_2 + \zeta_3)$$

We have the following:

(1) since $v(a^3\eta_1\eta_2) \ge 3v(a) + v(\eta_2) \ge 3e - 3 - e = 2e - 3 > 0$, then $a^3\eta_1\eta_2 \in R$;

(2) since $v(a^3\eta_3) \ge 2e - 3 > 0$, then $a^3\eta_3 \in R$;
Note that $v(a^3) = 3e - 3 \ge e$.

(3) since $v(a^3\eta_2) \ge 3e - 3 - e = 2e - 3 \ge e$, then $a^3\eta_2 \in I_{[\alpha]}$.

Moreover we see that $a^3\eta_1, a^3 \in I_{[\alpha]}$. Hence we have that $(a\zeta_1)^3 \in R\langle\alpha\rangle$ but $a\zeta_1 \notin R\langle\alpha\rangle$ because $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$. Since $a\zeta_1 \in R[\alpha]$, we conclude that $R\langle\alpha\rangle$ is not integrally closed in $R[\alpha]$, which is a contradiction.

Therefore we have $e \leq 2$, that is, $v(I_{[\alpha]}) = 1$ or 2, and consequently $v(\eta_2)$, $v(\eta_3)$, $v(\eta_4) = 0, -1, -2$, respectively.

Next suppose that $e = 2$. Take $a \in R$ such that $v(a) = e - 1 = 1$. We have

$$\begin{aligned}(a\zeta_1)^4 &= a^4((\eta_2(\eta_2 - \eta_1^2) - 2\eta_1\eta_3 - \eta_4) + (-2\eta_1\eta_2 - \eta_3 + \eta_1^3)\zeta_1 \\ &\quad + (3\eta_1^2 - \eta_2)\zeta_2 + 3\eta_1\zeta_3 + \zeta_4)\end{aligned}$$

Since $v(a) = 1$, $\eta_1 \in R$, and $v(\eta_2)$, $v(\eta_3)$, $v(\eta_4) \geq -2$, we have

$$\begin{aligned}v(a^4\eta_2^2) &\geq 4 + 2(-1) = 0, \\ v(a^4\eta_2\eta_1^2) &\geq 4 - 2 = 2 > 0, \\ v(a^4\eta_1\eta_3) &\geq 4 - 2 = 2 > 0, \\ v(a^4\eta_4) &\geq 4 - 2 = 2 > 0.\end{aligned}$$

Thus $a^4(\eta_2(\eta_2 - \eta_1^2) - 2\eta_1\eta_3 - \eta_4) \in R$.
Next we have:

$$\begin{aligned}v(a^4\eta_1) &\geq 4 - 2 = 2, \\ v(a^4\eta_3)4 - 2 &= 2, \\ v(a^4\eta_1^3) &\geq 4 > 2.\end{aligned}$$

Thus $a^4(-2\eta_1\eta_2 - \eta_3 + \eta_1^3) \in I_{[\alpha]}$.
Similarly, we have $a^4(3\eta_1^2 - \eta_2) \in I_{[\alpha]}$, $a^4\eta_1 \in I_{[\alpha]}$, and $a^4 \in I_{[\alpha]}$. Thus $(a\zeta_1)^4 \in R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ but $a\zeta_1 \notin R\langle\alpha\rangle$, which means that $R\langle\alpha\rangle$ is not integrally closed in $R[\alpha]$, a contradiction. Therefore we conclude that $e = 1$, that is, $I_{[\alpha]} = m$. ∎

Lemma 3.3.14 *Assume that R is a* DVR *with maximal ideal m. Let $v(\ )$ denote the valuation associated with R. Put $v(I_{[\alpha]}) = e$ and $v(I_{\eta_1}) = e - f$. If $I_{[\alpha]} \subseteq I_{\eta_1} \neq R$ and $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$, then $f = 0$ or 1.*

Proof Suppose that $f \geq 2$. Then $v(I_{\eta_1}) = e - f \geq 1$ implies that $e \geq 3$. Take $a \in R$ such that $v(a) = e - 1$. Then $a \notin I_{[\alpha]}$, $a\zeta_1 = a\alpha + a\eta_1 \in R[\alpha]$ and $a\zeta_1 \notin R\langle\alpha\rangle$. Consider the element

$$(a\zeta_1)^3 = a^3\zeta_1^3 = a^3(-(\eta_1\eta_2 + \eta_3) + (\eta_1^2 - \eta_2)\zeta_1 + 2\eta_1\zeta_2 + \zeta_3)$$

We have

$$v(a^3\eta_1\eta_2) = 3v(a) + v(\eta_1) + v(\eta_2) \geq 3e - (e - f) - e = e + f - 3 \geq 0$$

Thus $a^3\eta_1\eta_2 \in R$. We have

$$v(a^3\eta_3) = 3v(a) + v(\eta_3) \geq 3(e - 1) - e = 2e - 3 \geq 0.$$

Thus $a^3\eta_3 \in R$. We have

$$v(a^3\eta_1^2) = 3v(a) + 2v(\eta_1) = 3(e-1) - 2(e-f) = e + 2f - 3 \geq e.$$

$$v(a^3\eta_2) = 3v(a) + v(\eta_2) \geq 3(e-1) - e = 2e - 3 \geq e.$$

Thus $a^3\eta_1^2, a^3\eta_2 \in I_{[\alpha]}$. Similarly we have $a^3\eta_1, a^3 \in I_{[\alpha]}$. Therefore $(a\zeta_1)^3 \in R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$.

But $a\zeta_1 \notin R\langle\alpha\rangle$. So $R[\alpha] \cap R[\alpha^{-1}]$ is not integrally closed in $R[\alpha]$, a contradiction. ∎

Lemma 3.3.15 *Assume that R is a* DVR *with maximal ideal m. Let $v(\ )$ denote the valuation associated with R. Assume that $v(I_{[\alpha]}) \geq 2$. If $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$, then $v(I_{[\alpha]}) = v(I_{\eta_1})$.*

Proof Since $v(I_{[\alpha]}) \geq 2$, we have $v(I_{\eta_i}) \geq 2 - f \geq 1$, and hence $\eta_1 \notin R$ by Lemma 3.3.14, whence $v(I_{\eta_1}) \leq v(I_{[\alpha]}) \leq v(I_{\eta_1}) + 1$ by Lemma 3.3.14. Put $g = v(I_{\eta_1})$ and suppose that $v(I_{[\alpha]}) = g + 1$. Take $a \in m$ such that $v(a) = g$. Consider the element

$$(a\zeta_1)^3 = a^3\zeta_1^3 = a^3(-(\eta_1\eta_2 + \eta_3) + (\eta_1^2 - \eta_2)\zeta_1 + 2\eta_1\zeta_2 + \zeta_3)$$

Then

$$(a\zeta_1)^3 + (a^2\eta_2 - (a\eta_1)^2)(a\zeta_1) = a^3(-(\eta_1\eta_2 + \eta_3) + 2\eta_1\zeta_2 + \zeta_3)$$

Note here that $v(a^2\eta_2) \geq 2g - (g+1) = g - 1 \geq 0$ and that $v(a\eta_1) = v(a) + v(\eta_1) = g - g = 0$. Thus $a^2\eta_2 - (a\eta_1)^2 \in R$. Since $v(a^3\eta_1\eta_2) \geq 3g - g - (g+1) = g - 1 \geq 0$ and $v(a^3\eta_3) \geq 3g - (g+1) = 2g - 1 > 0$, we have $a^3(-(\eta_1\eta_2 + \eta_3)) \in R$. Since $v(a^3\eta_1) = 3g - g = 2g \geq g + 1$, we have $a^3\eta_1 \in I_{[\alpha]}$. Moreover since $v(a^3) = 3g \geq g + 1$, we have $a^3 \in I_{[\alpha]}$. Hence $(a\zeta_1)^3 + (a^2\eta_2 - (a\eta_1)^2)(a\zeta_1) \in R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$. So $a\zeta_1 \in R[\alpha]$ is integral over $R\langle\alpha\rangle$ but $a\zeta_1 \notin R\langle\alpha\rangle$, which is a contradiction. Hence $v(I_{[\alpha]}) = g = v(I_{\eta_1})$. ∎

The following theorem gives a perfect form of Proposition 3.2.18.

Theorem 3.3.16 *Assume that $\overline{R}$ is a finitely generated R-module and that the dimension formula holds between R and $\overline{R}$. Assume further that α is a super-primitive element of degree d over R. Then the following statements are equivalent:*

(i) *$R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$;*

(ii) *$I_{[\alpha]}$ is integrally closed and either $(I_{[\alpha]})_p = pR_p$ or $(I_{\eta_1})_p = (I_{[\alpha]})_p$ holds for each $p \in \mathrm{Ass}_R(R/I_{[\alpha]})$.*

Proof (ii) $\Rightarrow$ (i): For this implication, by Propositions 3.3.7 and 3.3.8 we need to show that

$$I_{[\alpha]} = \{a \in R | a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in R\}$$

We have only to show this equality by localizing at each prime divisor of $I_{[\alpha]}$. Since $I_{[\alpha]}$ is integrally closed, R_p is a DVR for each prime divisor p of $I_{[\alpha]}$ by Corollary 3.3.5. By the assumption, $(I_{[\alpha]})_p = pR_p$ or $(I_{\eta_1})_p = (I_{[\alpha]})_p$ for each $p \in \mathrm{Ass}_R(R/I_{[\alpha]})$. In the former case, the above equality holds because $I_{[\alpha]} \subseteq \{a \in R | a\eta_1, a^2\eta_2, \ldots, a^d\eta_d \in R\}$ is always valid. In the latter case, the condition $a \in R, a\eta_1, \ldots, a^d\eta_d \in R$ implies $a\eta_1 \in R$ and hence $a \in (I_{\eta_1})_p = (I_{[\alpha]})_p$ for each prime divisor p of $I_{[\alpha]}$.
(i) $\Rightarrow$ (ii) follows from Proposition 3.2.18. Since R_p is a DVR for each prime divisor of $I_{[\alpha]}$ by Corollary 3.3.5, the latter statement follows from Lemma 3.3.15. ∎

Corollary 3.3.17 *Assume that $\overline{R}$ is a finitely generated R-module and that the dimension formula holds between R and $\overline{R}$. Assume further that α is a super-primitive element of degree d over R and that $\eta_1 \in R$. Then $R\langle\alpha\rangle$ is integrally closed in $R[\alpha]$ if and only if $I_{[\alpha]}$ is integrally closed.*

Proof In this case, condition (ii) of Theorem 3.3.16 is that $I_{[\alpha]}$ is integrally closed and $(I_{[\alpha]})_p = pR_p$ for each $p \in \mathrm{Ass}_R(R/I_{[\alpha]})$. Hence $I_{[\alpha]}$ is a radical ideal. Thus Theorem 3.3.11 and Theorem 3.3.16 give rise to the equivalence. ∎

We end this section by showing that the matter mentioned in Theorem 3.3.16 is quite plain in the birational case. When $\alpha \in K$, that is, $R[\alpha]$ is a birational extension of R, the following result holds.

Proposition 3.3.18 *Assume that $\alpha \in K$ is a super-primitive element. Then R is integrally closed in $R[\alpha]$ if and only if $I_{[\alpha]} = I_\alpha$ is integrally closed.*

Proof ($\Leftarrow$): We have only to show that $N_{[\alpha]} \subseteq I_{[\alpha]}$. Take $a \in N_{[\alpha]}$. Then $a\alpha$ is integral over $R\langle\alpha\rangle = R$. Since α is super-primitive over R and since $I_{[\alpha]}$ is integrally closed, $a\alpha$ is anti-integral over R by Proposition 3.3.8. Thus $a\alpha$ is integral and anti-integral over R, and hence we have $a\alpha \in R$ by Remark 2.2.8. So we conclude that $a \in I_\alpha = I_{[\alpha]}$.

($\Rightarrow$): Since α is anti-integral over R, we have $R\langle\alpha\rangle = R$ by definition. Hence $I_{[\alpha]}$ is integrally closed by Proposition 3.2.18. ∎

3.4 Extensions of Type $R[\beta] \cap R[\beta^{-1}]$ with $\beta \in K(\alpha)$

Lemma 3.4.1 *Let a be an element of R. Then the following assertions hold:*

(1) $\varphi_{\alpha-a}(X) = \varphi_\alpha(X + a)$;

(2) $I_{[\alpha-a]} = I_{[\alpha]}$;

(3) *if α is an anti-integral element of degree d over R, then so is $\alpha - a$.*

Proof (1) For some $\lambda_1, \ldots, \lambda_d \in K$, we can write

$$\varphi_\alpha(X) = (X - a)^d + \lambda_1(X - a)^{d-1} + \cdots + \lambda_d$$

Then $\varphi_{\alpha-a}(X) = X^d + \lambda_1 X^{d-1} + \cdots \lambda_d = \varphi_\alpha(X + a)$.
(2) In the proof of the assertion (1), we get $I_{[\alpha-a]} = \bigcap_{i=1}^d (R :_R \lambda_i)$. Note that λ_i is an R-linear combination of $1, \eta_1, \ldots, \eta_i$ and η_i is an R-linear combination of $1, \lambda_1, \ldots, \lambda_i$. Hence it is easy to see that $\bigcap_{i=1}^d (R :_R \lambda_i) = \bigcap_{i=1}^d (R :_R \eta_i)$. Therefore $I_{[\alpha-a]} = I_{[\alpha]}$.
(3) Let $\pi' : R[X] \to R[\alpha - a] = R[\alpha]$ be the R-algebra homomorphism defined by $\pi'(X) = \alpha - a$. Take $f(X) \in R[X]$. Then we know the following implications:

$$\begin{aligned} f(X) \in \mathrm{Ker}(\pi') \quad &\Leftrightarrow \quad f(X - a) \in \mathrm{Ker}(\pi) \\ &\Leftrightarrow \quad f(X - a) \in I_{[\alpha]}\varphi_\alpha(X)R[X] \\ &\Leftrightarrow \quad f(X) \in I_{[\alpha]}\varphi_\alpha(X + a)R[X] \\ &\Leftrightarrow \quad f(X) \in I_{[\alpha-a]}\varphi_{\alpha-a}(X)R[X] \end{aligned}$$

Hence $\mathrm{Ker}(\pi') = I_{[\alpha-a]}\varphi_{\alpha-a}(X)R[X]$, which implies that $\alpha - a$ is an anti-integral element of degree d over R. ∎

Lemma 3.4.2 *Assume that α is an anti-integral element of degree d over R. Let a be a nonzero element of R. Then α is also an anti-integral element of degree d over $R[1/a]$.*

Proof Note that $R[1/a]$ is flat over R. By tensoring the exact sequence:

$$0 \to I_{[\alpha]}\varphi_\alpha(X)R[X] \to R[X] \to R[\alpha] \to 0$$

we have an exact sequence:

$$0 \to I_{[\alpha]}\varphi_\alpha(X)R[1/a][X] \to R[1/a][X] \to R[1/a][\alpha] \to 0$$

Besides, we know that

$$I_{[\alpha]}R[1/a] = \bigcap_{i=1}^{d}(R[1/a] :_{R[1/a]} \eta_i)$$

because $R[1/a]$ is flat over R. Hence α is also an anti-integral element of degree d over $R[1/a]$. ∎

Lemma 3.4.3 *Assume that α is a nonzero anti-integral element of degree d over R. Put $A := R[\alpha]$. Then the following two assertions hold:*

(1) $I_{[\alpha^{-1}]} = I_{[\alpha]}\eta_d = I_{[\alpha]}\varphi_\alpha(0)$;

(2) $I_{[\alpha^{-1}]} = \alpha A \cap R$.

Proof (1) It is easily verified that $\varphi_{\alpha^{-1}}(X) = X^d + \eta_d^{-1}\eta_{d-1}X^{d-1} + \cdots + \eta_d^{-1}$ and $I_{[\alpha^{-1}]} = \bigcap_{i=1}^{d-1}(R :_R \eta_d^{-1}\eta_i) \cap (R :_R \eta_d^{-1})$. Let a be an element of $I_{[\alpha^{-1}]}$. Since a is in $(R :_R \eta_d^{-1})$, there exists an element b of R such that $a = \eta_d b$. We see that $b\eta_i = \eta_d^{-1}a\eta_i \in R$ for $i = 1, \ldots, d-1$ and $b\eta_d = a \in R$. Thus b is in $I_{[\alpha]}$. So $a = b\eta_d \in I_{[\alpha]}\eta_d$. Hence we conclude that $I_{[\alpha^{-1}]} \subseteq I_{[\alpha]}\eta_d$. Conversely, assume that a is in $I_{[\alpha]}\eta_d$. Then we can take an element b of $I_{[\alpha]}$ such that $a = b\eta_d$. Note that a is in R. Since $a\eta_d^{-1}\eta_i = b\eta_i \in R$ for $i = 1, \ldots, d-1$ and $a\eta_d^{-1} = b \in R$, we know that a is in $I_{[\alpha^{-1}]}$. Hence $I_{[\alpha]}\eta_d \subseteq I_{[\alpha^{-1}]}$.
(2) By assertion (1), we have $I_{[\alpha]}\eta_d = I_{[\alpha^{-1}]}$. So we have only to prove that $\alpha A \cap R \supseteq I_{[\alpha]}\eta_d$. Take a in $I_{[\alpha]}$. Then $\varphi_\alpha(\alpha) = 0$ implies that $-\alpha(a\alpha^{d-1} + a\eta_1\alpha^{d-2} + \cdots + a\eta_{d-1}) = a\eta_d$. Hence $a\alpha^{d-1} + a\eta_1\alpha^{d-2} + \cdots + a\eta_{d-1}$ belongs to A and $a\eta_d \in R$. Hence $a\eta_d$ is in $\alpha A \cap R$, and so $\alpha A \cap R \supseteq I_{[\alpha]}\eta_d = I_{[\alpha^{-1}]}$. Now suppose that a is in $\alpha A \cap R$. Then we can write $-a = \alpha(a_1 + a_2\alpha + \cdots + a_n\alpha^{n-1})$ for some $a_1, \ldots, a_n \in R$ and a positive integer n. Set $f(X) := a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a$. Then $f(X)$ is in $\mathrm{Ker}(\pi)$. Therefore there exist elements $h_i(X) \in I_{[\alpha]}\varphi_\alpha(X)$ and $g_i \in R[X]$ such that $f(X) = \sum h_i(X)g_i(X)$. This shows that $a = f(0) = \sum h_i(0)g_i(0) \in I_{[\alpha]}\eta_d = I_{[\alpha^{-1}]}$ because $h_i(0) \in I_{[\alpha]}\eta_d$ and $g_i(0) \in R$. ∎

The following result is a generalization of Lemma 3.4.3.

Proposition 3.4.4 *Assume that α is an anti-integral element of degree d over R. Put $A := R[\alpha]$. Let a be an element of R such that $\alpha - a$ is not zero. Then*

$$I_{[(\alpha-a)^{-1}]} = I_{[\alpha]}\varphi_\alpha(a) = (\alpha - a)A \cap R$$

Proof By Lemma 3.4.1(3), we have that $\alpha - a$ is an anti-integral element of degree d over R. Hence Lemma 3.4.3(2) yields that $I_{[(\alpha-a)^{-1}]} = (\alpha - a)A \cap R$ because $R[\alpha - a] = R[\alpha]$. On the other hand, Lemma 3.4.3(1) implies that $I_{[(\alpha-a)^{-1}]} = I_{[\alpha-a]}\varphi_{\alpha-a}(0)$. By Lemmas 3.4.3(1) and 3.4.1(2), we get $I_{[\alpha-a]} = I_{[\alpha]}$ and $\varphi_{\alpha-a}(0) = \varphi_\alpha(a)$. Thus $I_{[(\alpha-a)^{-1}]} = I_{[\alpha]}\varphi_\alpha(a)$. ∎

Corollary 3.4.5 *Under the same assumptions as in Proposition 3.4.4, $\alpha - a$ is a unit of A if and only if $I_{[\alpha]}\varphi_\alpha(a) = R$.*

Proof We can easily see that $\alpha - a$ is a unit of A if and only if $(\alpha - a)A \cap R = R$. Hence we reach the conclusion by Proposition 3.4.4. ∎

Lemma 3.4.6 *Assume that R is a Noetherian domain and that α is a nonzero anti-integral element of degree d over R. Let a be a nonzero element of R. Then the following assertions hold:*

(1) *α^{-1} is an anti-integral element of degree d over R;*

(2) *α^{-1} is an anti-integral element of degree d over $R[1/a]$.*

Proof (1) is referred to Theorem 3.4.5. (2) is immediate from the assertion (1) and Lemma 3.4.2. ∎

Lemma 3.4.7 *Assume that R is a Noetherian domain and that α is a nonzero anti-integral element of degree d over R. Let a be a nonzero element of R. Then the following conditions are equivalent to each other:*

(i) *$\alpha^{-1} \in R[1/a][\alpha]$;*

(ii) *α^{-1} is integral over $R[1/a]$;*

(iii) *$a \in \sqrt{I_{[\alpha^{-1}]}}$.*

Proof (i) $\Rightarrow$ (ii): We can write

$$\alpha^{-1} = b_0 + b_1\alpha + \cdots + b_n\alpha^n \qquad (1)$$

for some $b_1, \ldots, b_n \in R[1/a]$. Dividing both sides of equality (1) by α^n, we get

$$(\alpha^{-1})^{n+1} = b_0(\alpha^{-1})^n + b_1(\alpha^{-1})^{n-1} + \cdots + b_n$$

This shows that α^{-1} is integral over $R[1/a]$.
(ii) $\Rightarrow$ (iii): Assume that α^{-1} is integral over $R[1/a]$. Then there exists a monic polynomial $f(X)$ of $R[1/a][X]$ such that $f(\alpha^{-1}) = 0$. By Lemma 3.4.6(2), we

see that α^{-1} is an anti-integral element of degree d over $R[1/a]$. Hence $f(X)$ is in $I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(X)R[1/a][X]$. Considering the leading coefficient of $f(X)$, we see that $1 = c/a^n$ for some $c \in I_{[\alpha^{-1}]}$ and a nonnegative integer n. If $n > 0$, then a is in $\sqrt{I_{[\alpha^{-1}]}}$. If $n = 0$, then $c = 1$ and $I_{[\alpha^{-1}]} = R$. Again we have $a \in \sqrt{I_{[\alpha^{-1}]}}$.
(iii) $\Rightarrow$ (i): Suppose that $a \in \sqrt{I_{\alpha^{-1}]}}$. Then there exists a positive integer r such that $a^r \in I_{[\alpha^{-1}]}$. Write $\varphi_{\alpha^{-1}}(X) := X^d + \mu_1 X^{d-1} + \cdots + \mu_d$, where $\mu_1, \ldots, \mu_d \in K$. Thus, we have

$$\alpha^{-1} = -(\mu_1 + \mu_2\alpha + \cdots + \mu_d\alpha^{d-1})$$

Hence

$$a^r\alpha^{-1} = -(a^r\mu_1 + a^r\mu_2\alpha + \cdots a^r\mu_d\alpha^{d-1})$$

and $a^r\mu_1, a^r\mu_2, \ldots, a^r\mu_d$ are in R. This means that α^{-1} is in $R[1/a][\alpha]$. $\square$

Proposition 3.4.8 *Assume that R is a Noetherian domain and that α is a nonzero anti-integral element of degree d over R. Let a be an element of R such that $\alpha - a$ is not zero. Then $R[\alpha^{-1}] \subseteq R[(\alpha - a)^{-1}]$ if and only if $a \in \sqrt{I_{[\alpha]}\varphi_\alpha(0)}$.*

Proof ($\Rightarrow$): We can write

$$\alpha^{-1} = a_0 + a_1(\alpha - a)^{-1} + \cdots + a_n(\alpha - a)^{-1} \qquad (1)$$

for some $a_0, a_1, \ldots, a_n \in R$. There exists a polynomial $g(X) \in R[X]$ such that $X^n = (X + a)g(X) + (-a)^n$. Then we have $(\alpha - a)^n = \alpha g(\alpha - a) + (-a)^n$ by substituting $\alpha - a$ for X. Multiplying both sides of equality (1) by $(\alpha - a)^n$, we know that $\alpha^{-1}(\alpha - a)^n = g(\alpha - a) + (-a)^n\alpha^{-1}$ belongs to $R[\alpha - a] = R[\alpha]$. Hence $(-a)^n\alpha^{-1}$ is in $R[\alpha]$, and so α^{-1} belongs to $R[1/a][\alpha]$. Thus we have that $a \in \sqrt{I_{[\alpha^{-1}]}} = \sqrt{I_{[\alpha]}\varphi_\alpha(0)}$ by Lemmas 3.4.6 and 3.4.7.
($\Leftarrow$): Lemma 3.4.7 asserts that α^{-1} is in $R[1/a][\alpha = R[1/a][\alpha - a]$. So there exist elements $b_0, b_1, \ldots, b_n \in R$ such that $(-a)^n\alpha^{-1} = b_0 + b_1(\alpha - a) + \cdots + b_n(\alpha - a)^n$. Note that b_n may be zero. Moreover, there exists a polynomial $g(X) \in R[X]$ satisfying $X^n = (X + a)g(X) + (-a)^n$ with $\deg(g(X)) \leq n - 1$. Put $\beta := \alpha - a$. Then we have $(-a)^n = \beta^n - \alpha g(\beta)$ and

$$(-a)^n\alpha^{-1} = b_0 + b_1\beta + \cdots + b_n\beta^n \qquad (2)$$

Substituting $\beta^n - \alpha g(\beta)$ for $(-a)^n$ in the equality (2), we can write $\beta^n\alpha^{-1} = a_0 + a_1\beta + \cdots + a_n\beta^n$ for some $a_0, a_1, \ldots, a_n \in R$. Hence $\alpha^{-1} = a_0\beta^{-n} + a_1\beta - (n-1) + \cdots + a_n$ is in $R[\beta^{-1}] = R[(\alpha - a)^{-1}]$. This shows that $R[\alpha^{-1}] \subseteq R[(\alpha - a)^{-1}]$. $\square$

Theorem 3.4.9 *Assume that R is a Noetherian domain and that α is a nonzero anti-integral element of degree d over R. Let a be an element of R such that*

$\alpha - a$ is not zero. Then $R[\alpha^{-1}] = R[(\alpha - a)^{-1}]$ if and only if $a \in \sqrt{I_{[\alpha]}\varphi_\alpha(0)} \cap \sqrt{I_{[\alpha]}\varphi_\alpha(a)}$.

Proof ($\Rightarrow$): Since $R[\alpha^{-1}] \subseteq R[(\alpha - a)^{-1}]$, we see that a is in $\sqrt{I_{[\alpha]}\varphi_\alpha(0)}$ by Proposition 3.4.8. On the other hand, we have $R[(\alpha - a)^{-1}] \subseteq R[\alpha^{-1}]$. By Lemma 3.4.1, we know that $\alpha - a$ is an anti-integral element of degree d over R and that $I_{[\alpha-a]}\varphi_{\alpha-a}(0) = I_{[\alpha]}\varphi_\alpha(a)$. Thus Proposition 3.4.8 yields that $-a$ is in $\sqrt{I_{[\alpha-a]}\varphi_{\alpha-a}(0)}$. So we have the assertion.
($\Leftarrow$): By Proposition 3.4.8, we get $R[\alpha^{-1}] \subseteq R[(\alpha - a)^{-1}]$ because a is in $\sqrt{I_{[\alpha]}\varphi_\alpha(0)}$. Similarly, we have $R[(\alpha - a)^{-1}] \subseteq R[\alpha^{-1}]$ because $-a$ is in $\sqrt{I_{[\alpha]}\varphi_\alpha(a)} = \sqrt{I_{[\alpha-a]}\varphi_{\alpha-a}(0)}$. So we complete the proof. ∎

Lemma 3.4.10 *Assume that α is an anti-integral element of degree d over R. Then*

$$\{p \in \mathrm{Spec}(R) \mid \alpha \text{ is integral over } R_p\} = \{p \in \mathrm{Spec}(R) \mid I_{[\alpha]} \not\subseteq p\}$$

Proof Assume that $I_{[\alpha]} \not\subseteq p$. Then there exists an element a of $I_{[\alpha]}$ such that $a \notin p$. Hence $a\eta_1, \ldots, a\eta_d$ are in R, and $\eta_1, \ldots, \eta_d$ are in R_p. Therefore α is integral over R_p because $\varphi_\alpha(\alpha) = 0$. Conversely, assume that α is integral over R_p. Then there exists a monic polynomial $f(X)$ in $R_p[X]$ such that $f(\alpha) = 0$. This implies that $f(X)$ is in $I_{[\alpha]}\varphi_{[\alpha]}\varphi_\alpha(X)R_p[X]$ because α is an anti-integral element of degree d over R. Considering the leading coefficient of $f(X)$, we see that $I_{[\alpha]}R_p = R_p$. Hence $I_{[\alpha]} \not\subseteq p$. ∎

Proposition 3.4.11 *Assume that R is a Noetherian domain and that α is a nonzero anti-integral element of degree d over R. Let a be an element of R such that $\alpha - a$ is not zero. Then the following two conditions are equivalent to each other:*

(i) $a \in \sqrt{I_{[\alpha]}\varphi_\alpha(0)} \cap \sqrt{I_{[\alpha]}\varphi_\alpha(a)}$;
(ii) $a \in \sqrt{I_{[\alpha]}\varphi_\alpha(0)} = \sqrt{I_{[\alpha]}\varphi_\alpha(a)}$.

Proof (ii) $\Rightarrow$ (i) is obvious.
(i) $\Rightarrow$ (ii): By Theorem 3.4.9, we see that $R[\alpha^{-1}] = R[(\alpha - a)^{-1}]$. Lemma 3.4.10 shows that the obstruction ideal of integrality of the extension $R[\alpha^{-1}]$ of R is $\sqrt{I_{[\alpha^{-1}]}} = \sqrt{I_{[\alpha]}\varphi_\alpha(0)}$. Also the obstruction ideal of integrality of the extension $R[(\alpha - a)^{-1}]$ of R is $\sqrt{I_{[(\alpha-a)^{-1}]}} = \sqrt{I_{[\alpha]}\varphi_\alpha(a)}$. Hence $\sqrt{I_{[\alpha]}\varphi_\alpha(0)} = \sqrt{I_{[\alpha]}\varphi_\alpha(a)}$. ∎

Recall that $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_{d-1}X + \eta_d$.

Proposition 3.4.12 *Let (R, m) be a Noetherian local domain and assume that α is an anti-integral element of degree $d \geq 2$ over R. Assume that $I_{[\alpha]}\eta_{d-1} \subseteq m$. Then for every $a \in I_{[\alpha]}\eta_d \cap m$, the following hold:*

(1) $R[\alpha^{-1}] = R[(\alpha - a)^{-1}]$;

(2) $R[\alpha] \cap R[\alpha^{-1}] = R[\alpha - a] \cap R[(\alpha - a)^{-1}]$.

Proof (1) Note that $\alpha - a \neq 0$ because $d \geq 2$. Since a is in $I_{[\alpha]}\eta_d$, there exists an element b of $I_{[\alpha]}$ with $a = b\eta_d$. Set $a_i = b\eta_i$ for $i = 1, \ldots, d-1$ and $a_0 = b$. Then $b\varphi_\alpha(a) = a_0a^d + a_1a^{d-1} + \cdots + a_{d-1}a + a = a(a_0a^{d-1} + \cdots + a_{d-1} + 1)$. Put $c := a_0a^{d-1} + \cdots + a_{d-1} + 1$. Then c is a unit of R because a and a_{d-1} are in m. Thus $a = c^{-1}b\varphi_\alpha(a)$ is in $I_{[\alpha]}\varphi_\alpha(a)$. Hence a is in $I_{[\alpha]}\eta_d = I_{[\alpha]}\varphi_\alpha(0) \cap I_{[\alpha]}\varphi_\alpha(a)$. Therefore Theorem 3.4.9 implies that $R[\alpha^{-1}] = R[(\alpha - a)^{-1}]$.
(2) is clear from the assertion (1). ∎

Lemma 3.4.13 *Let a be an element of R. If $\alpha(\neq 0)$ is integral over R, then*

$$R[\alpha] \cap R[\alpha^{-1}] = R[\alpha - a] \cap R[(\alpha - a)^{-1}]$$

Proof Since α is integral over R, there exist elements $a_1, \ldots, a_n$ of R such that $\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0$. Hence $\alpha = -(a_1 + a_2\alpha^{-1} + \cdots + a_n\alpha^{-(n-1)})$ belongs to $R[\alpha^{-1}]$. By using the fact that $\alpha - a$ is integral over R, we also get $R[\alpha - a] \cap R[(\alpha - a)^{-1}] = R[\alpha - a] = R[\alpha]$. We come to the conclusion. ∎

Theorem 3.4.14 *Assume that R is a Noetherian domain and that α is an anti-integral element of degree $d \geq 2$ over R. Assume that $I_{[\alpha]}\eta_{d-1} \subseteq I_{[\alpha]}$. Then for every element $a \in \sqrt{I_{[\alpha]}} \cap I_{[\alpha]}\eta_d$, the following holds:*

$$R[\alpha] \cap R[\alpha^{-1}] = R[\alpha - a] \cap R[(\alpha - a)^{-1}]$$

Proof Take $p \in \mathrm{Spec}(R)$. If $I_{[\alpha]} \not\subseteq p$, then α is integral over R_p by Lemma 3.4.10. Thus we have $R_p[\alpha] \cap R_p[\alpha^{-1}] = R_p[\alpha - a] \cap R_p[(\alpha - a)^{-1}]$ by Lemma 3.4.13. If $I_{[\alpha]} \subseteq p$, then we have $R_p[\alpha] \cap R_p[\alpha^{-1}] = R_p[\alpha - a] \cap R_p[(\alpha - a)^{-1}]$ by Proposition 3.4.12(2) and the assumption $I_{[\alpha]}\eta_{d-1} \subseteq I_{[\alpha]}$. Therefore $R[\alpha] \cap R[\alpha^{-1}] = R[\alpha - a] \cap R[(\alpha - a)^{-1}]$. ∎

We will generalize Theorem 3.4.9 as follows.

Theorem 3.4.15 *Assume that R is a Noetherian domain and that α is an anti-integral element of degree d over R. Let a and b be elements of R such*

that $\alpha - a \neq 0$ *and* $\alpha - b \neq 0$. *Then* $R[(\alpha - a)^{-1}] = R[(\alpha - b)^{-1}]$ *if and only if* $a - b \in \sqrt{I_{[\alpha]}\varphi_\alpha(a)} = \sqrt{I_{[\alpha]}\varphi_\alpha(b)}$.

Proof Put $\beta := \alpha - a$. Then $\alpha - b = \beta - (b - a)$. By Lemma 3.4.1(3), we see that β is also an anti-integral element of degree d over R. Hence Theorem 3.4.9 asserts that $R[\beta^{-1}] = R[(\beta - (b - a))^{-1}]$ if and only if $b - a \in \sqrt{I_{[\beta]}\varphi_\beta(0)} \cap \sqrt{I_{[\beta]}\varphi_\beta(b - a)}$. By Lemma 3.4.1, we see that $\varphi_\beta(X) = \varphi_\alpha(X + a)$ and $I_{[\beta]} = I[\alpha]$. Hence $\varphi_\beta(0) = \varphi_\alpha(a)$ and $\varphi_\beta(b - a) = \varphi_\alpha(b)$. Similarly, by using Proposition 3.4.11, we can easily verify that $a - b \in \sqrt{I_{[\alpha]}\varphi_\alpha(a)} \cap \sqrt{I_{[\alpha]}\varphi_\alpha(b)}$ if and only if $a - b \in \sqrt{I_{[\alpha]}\varphi_\alpha(a)} = \sqrt{I_{[\alpha]}\varphi_\alpha(b)}$. So we get the required conclusion. ∎

Next we show that the equality $R[\alpha - a] \cap R[(\alpha - a)^{-1}] = R[\alpha] \cap R[\alpha^{-1}]$ holds for any $a \in R$ with $\alpha - a \neq 0$.

Lemma 3.4.16 *Let a be an element of R. Then the following assertions hold:*

(1) $\varphi_{\alpha-a}(X) = \varphi_\alpha(X + a)$;

(2) $I_{[\alpha-a]} = I_{[\alpha]}$;

(3) *if* α *is an anti-integral element of degree d over R, then so is* $\alpha - a$.

Proof (1) For some $\lambda_1, \ldots, \lambda_d \in K$, we can write

$$\varphi_\alpha(X) = (X - a)^d + \lambda_1(X - a)^{d-1} + \cdots + \lambda_d$$

Then $\varphi_{\alpha-a}(X) = X^d + \lambda_1 X^{d-1} + \cdots \lambda_d = \varphi_\alpha(X + a)$.
(2) In the proof of the assertion (1), we get $I_{[\alpha-a]} = \bigcap_{i=1}^d (R :_R \lambda_i)$. Note that λ_i is an R-linear combination of $1, \eta_1, \ldots, \eta_i$ and η_i is an R-linear combination of $1, \lambda_1, \ldots, \lambda_i$. Hence it is easy to see that $\bigcap_{i=1}^d (R :_R \lambda_i) = \bigcap_{i=1}^d (R :_R \eta_i)$. Therefore $I_{[\alpha-a]} = I_{[\alpha]}$.
(3) Let $\pi' : R[X] \to R[\alpha - a] = R[\alpha]$ be the R-algebra homomorphism defined by $\pi'(X) = \alpha - a$. Take $f(X) \in R[X]$. Then we know the following implications: $f(X) \in \mathrm{Ker}(\pi') \Leftrightarrow f(X - a) \in \mathrm{Ker}(\pi) \Leftrightarrow f(X - a) \in I_{[\alpha]}\varphi_\alpha(X)R[X] \Leftrightarrow f(X) \in I_{[\alpha]}\varphi_\alpha(X+a)R[X] \Leftrightarrow f(X) \in I_{[\alpha-a]}\varphi_{\alpha-a}(X)R[X]$. Hence $\mathrm{Ker}(\pi') = I_{[\alpha-a]}\varphi_{\alpha-a}(X)R[X]$, which implies that $\alpha - a$ is an anti-integral element of degree d over R. ∎

Let $\zeta_i := \alpha^i + \eta_1\alpha^{i-1} + \eta_2\alpha^{i-2} + \cdots + \eta_i \in K[\alpha]$ $(1 \leq i \leq d - 1)$ and put $\Delta_i := I_{[\alpha]}\zeta_i$ $(1 \leq i \leq d-1)$, which is an R-module. When R is Noetherian, the following result is seen in Theorem 3.1.18. But the same result is valid for any integral domains and the same proof is effective. We show it for convenience sake.

Lemma 3.4.17 *Assume that α is an anti-integral element of degree d over R. Then $R[\alpha] \cap R[\alpha^{-1}] = R \oplus \Delta_1 \oplus \cdots \oplus \Delta_{d-1} = R \oplus (\bigoplus_{i=1}^{d-1} I_{[\alpha]}\zeta_i)$, as R-modules.*

Proof We shall show that $R[\alpha] \cap R[\alpha^{-1}] = R + \Delta_1 + \cdots + \Delta_{d-1}$, as R-modules. The inclusion $R + \Delta_1 + \cdots + \Delta_{d-1} \subseteq R[\alpha] \cap R[\alpha^{-1}]$ is obvious. We must show the converse inclusion. Take $\beta \in R[\alpha] \cap R[\alpha^{-1}]$ and put $\beta = a_0\alpha^n + \cdots + a_n = b_0(\alpha^{-1})^m + \cdots + b_m$ with $a_i, b_j \in R$. We prove our assertion by induction on m. If $m = 0$, then $\beta = b_0 \in R$. First we will show the case $m < d$. Put $f(X) = a_0X^{n+m} + \cdots + a_nX^m - b_mX^m - \cdots - b_0 \in R[X]$. Then $f(\alpha) = 0$ and hence $f(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$. Thus $f(0) = -b_0 \in I_{[\alpha]}\eta_d$, so that there exist $c_0, \ldots, c_d \in R$ with $c_d = b_0$ such that $c_0\alpha^d + c_1\alpha^{d-1} + \cdots + c_d = 0$. Indeed, since $b_0 = a\eta_d$ for some $a \in I_{[\alpha]}$, we may put $c_0 = a, c_i = a\eta_i$ $(1 \le i \le d)$. Since $m < d$, putting $\gamma = c_d(\alpha^{-1})^m + \cdots + c_{d-m+1} = -(c_{d-m} + c_{d-m-1}\alpha + \cdots + c_0\alpha^{d-m})$, we see that $\gamma \in \Delta_{d-m}$ and the element $\beta - \gamma$ satisfies our induction hypothesis. Since $\gamma \in \Delta_{d-m}$, it follows that $\beta \in R + \Delta_1 + \cdots + \Delta_{d-1}$. Secondly we consider the case $m \ge d$. As the above statements, there exists $c_0, \ldots, c_d \in R$ with $c_d = b_0$ such that $c_0\alpha^d + c_1\alpha^{d-1} + \cdots + c_d = 0$. Put $\gamma := c_d(\alpha^{-1}) = -(c_0\alpha^{d-1} + \cdots + c_{d-1}) \in \Delta_{d-1}$. Then $b_0(\alpha^{-1})^m = \gamma(\alpha^{-1})^m$. Hence we have $\beta = \gamma(\alpha^{-1})^{m-1} + \cdots + b_m = -(c_0\alpha^{d-1} + \cdots + c_{d-1})(\alpha^{-1})^{m-1} + b_1(\alpha^{-1})^{m-1} + \cdots b_m = (b_1 - c_{d-1})(\alpha^{-1})^{m-1} + (b_2 - c_{d-2})(\alpha^{-1})^{m-2} + \cdots + b_m$, which satisfies the induction hypothesis. Thus $\beta \in R + \Delta_1 + \cdots + \Delta_{d-1}$. Hence we have proved that $R[\alpha] \cap R[\alpha^{-1}] = R + \Delta_1 + \cdots + \Delta_{d-1}$. Since $(R[\alpha] \cap R[\alpha^{-1}]) \otimes_R K = K(\alpha)$ is a d-dimensional vector space over K, it follows that $R[\alpha] \cap R[\alpha^{-1}]$ is an R-module of rank d. Hence $R[\alpha] \cap R[\alpha^{-1}] = R \oplus \Delta_1 \oplus \cdots \oplus \Delta_{d-1} = R \oplus (\bigoplus_{i=1}^{d-1} I_{[\alpha]}\zeta_i)$ as R-modules. $\square$

Let $f : \mathbf{Z} \to R$ be a ring homomorphism sending $n \in \mathbf{Z}$ to $n \cdot 1 \in R$, where $\mathbf{Z}$ denotes the ring of integers. Let $F := \mathbf{Z}/\mathrm{Ker}(f)$, which is a subdomain of R.

Let a be an element of R such that $\alpha - a \neq 0$. Then $\alpha - a$ is of degree d over R. Let $\varphi_{\alpha-a}(X) := X^d + \eta'_1X^{d-1} + \cdots + \eta'_d$ be the monic minimal polynomial of $\alpha - a$ over K, where $\eta'_1, \ldots, \eta'_d \in K$. Put $\zeta'_i := (\alpha - a)^i + \eta'_1(\alpha - a)^{i-1} + \eta'_2(\alpha - a)^{i-2} + \cdots + \eta'_i \in K[\alpha - a]$ $(1 \le i \le d - 1)$. Then putting $X := Y + a$, we have $\varphi_\alpha(X) = \varphi_\alpha(Y + a) = (Y + a)^d + \eta_1(Y + a)^{d-1} + \cdots + \eta_d = Y^d + \eta'_1Y^{d-1} + \cdots + \eta'_d = \varphi_{\alpha-a}(Y)$ (*cf.* Lemma 3.4.1(1)). Note that $\varphi_\alpha(a) = \eta'_d$. Put $\beta := \alpha - a$. Then we have for each $k = 1, \ldots, d - 2$,

$$\begin{aligned} \zeta'_{k+1} &= \beta\zeta'_k + \eta'_{k+1} \\ \zeta_{k+1} &= \alpha\zeta_k + \eta_{k+1}, \quad \text{i.e., } \alpha\zeta_k = \zeta_{k+1} - \eta_{k+1} \end{aligned}$$

Comparing the coefficients of $\varphi_\alpha(Y+a) = \varphi_{\alpha-a}(Y)$, we have

$$\begin{aligned}\eta'_k &= \binom{d}{k}a^k + \eta_1\binom{d-1}{k-1}a^{k-1} + \eta_2\binom{d-2}{k-2}a^{k-2} \\ &\quad + \cdots + \eta_{k-1}\binom{d-k+1}{1}a + \eta_k\end{aligned}$$

for $k = 1, \ldots, d-1$. So we have $\eta'_k \in F[a] + F[a]\eta_1 + \cdots + F[a]\eta_{d-1}$ for each k $(1 \le k \le d-1)$.

Proposition 3.4.18 *Under the notation as above, for each k $(1 \le k \le d-1)$,*

$$\zeta'_k - \zeta_k \in F[a]\zeta_{k-1} + \cdots + F[a]\zeta_1 + F[a] + F[a]\eta_1 + \cdots + F[a]\eta_{d-1}$$

Proof We shall prove this equality by induction on k.

Case (I) $k = 1$:

$$\begin{aligned}\zeta'_1 &= \beta + \eta'_1 \\ &= \alpha + a + \eta'_1 \\ &= \alpha + \eta_1 + (\eta'_1 - a - \eta_1) \\ &= \zeta_1 + (\eta'_1 - a - \eta_1)\end{aligned}$$

Hence $\zeta'_1 - \zeta_1 = \eta'_1 - a - \eta_1 \in F[a] + F[a]\eta_1$.

Case (II) Assume that the equality holds for $k(\ge 1)$.

That is, there exist $\lambda_i(a) \in F[a]$ $(1 \le i \le k-1)$ and $\lambda_0(a) \in F[a] + F[a]\eta_1 + \cdots + F[a]\eta_{d-1}$ such that $\zeta'_k = \zeta_k + \lambda_{k-1}(a)\zeta_{k-1} + \cdots + \lambda_1(a)\zeta_1 + \lambda_0(a)$. Then

$$\begin{aligned}\zeta'_{k+1} &= \beta\zeta'_k + \eta'_{k+1} \\ &= (\alpha - a)(\zeta_k + \lambda_{k-1}(a)\zeta_{k-1} + \lambda_{k-2}(a)\zeta_{k-2}(a) + \cdots + \lambda_0(a)) \\ &= \alpha\zeta_k + \lambda_{k-1}(a)\alpha\zeta_{k-1} + \lambda_{k-2}(a)\alpha\zeta_{k-2} + \cdots + \lambda_0(a)\alpha - a\zeta_k \\ &\quad - a\lambda_{k-1}(a)\zeta_{k-1} - a\lambda_{k-2}(a)\zeta_{k-2} - \cdots - a\lambda_0(a) + \eta'_{k+1} \\ &= (\zeta_{k+1} - \eta_{k+1}) + \lambda_{k-1}(a)(\zeta_k - \eta_k) + \cdots + \lambda_1(a)(\zeta_2 - \eta_2) \\ &\quad + \lambda_0(a)(\zeta_1 - \eta_1) - a\zeta_k - a\lambda_{k-1}(a)\zeta_{k-1} - \cdots - a\lambda_1(a)\zeta_1 \\ &\quad - a\lambda_0(a) + \eta'_{k+1}\end{aligned}$$

$$\begin{aligned} = \quad & \zeta_{k+1} + (\lambda_{k-1}(a) - a)\zeta_k + (\lambda_{k-2}(a) - a\lambda_{k-1}(a))\zeta_{k-1} + \cdots \\ & + (\lambda_0(a) - a\lambda_1(a))\zeta_1 - (\eta_{k+1} + \lambda_{k-1}(a)\eta_k + \cdots + \lambda_0(a)\eta_1 \\ & + a\lambda_0(a)) + \eta'_{k+1} \end{aligned}$$

Thus $\zeta'_{k+1} - \zeta_{k+1} \in F[a]\zeta_k + F[a]\zeta_{k-1} + \cdots + F[a]\zeta_1 + F[a] + F[a]\eta_1 + \cdots + F[a]\eta_{d-1}$. Hence we are done. ∎

Under these preparations, we have the following result.

Theorem 3.4.19 *Let R be an integral domain and let α be an anti-integral element of degree d over R. Then for any $a \in R$ with $\alpha - a \neq 0$,*

$$R[\alpha] \cap R[\alpha^{-1}] = R[\alpha - a] \cap R[(\alpha - a)^{-1}]$$

Proof We see that $\alpha - a$ is an anti-integral element of degree d over R and that $I_{[\alpha]} = I_{[\alpha-a]}$ by Lemma 3.4.1. Hence $R[\alpha] \cap R[\alpha^{-1}] = R + I_{[\alpha]}\zeta_1 + \cdots + I_{[\alpha]}\zeta_{d-1}$ and $R[\alpha - a] \cap R[(\alpha - a)^{-1}] = R + I_{[\alpha-a]}\zeta'_1 + \cdots + I_{[\alpha-a]}\zeta'_{d-1}$ by Lemma 3.4.17. By Proposition 3.4.18, we have $R + I_{[\alpha-a]}\zeta'_1 + \cdots + I_{[\alpha-a]}\zeta'_{d-1} = R + I_{[\alpha]}\zeta'_1 + \cdots + I_{[\alpha]}\zeta'_{d-1} \subseteq R + I_{[\alpha]}\zeta_1 + \cdots + I_{[\alpha]}\zeta_{d-1}$, whence $R[\alpha - a] \cap R[(\alpha - a)^{-1}] \subseteq R[\alpha] \cap R[\alpha^{-1}]$. By symmetry, we have $R[\alpha] \cap R[\alpha^{-1}] \subseteq R[\alpha - a] \cap R[(\alpha - a)^{-1}]$. Therefore $R[\alpha] \cap R[\alpha^{-1}] = R[\alpha - a] \cap R[(\alpha - a)^{-1}]$. ∎

Example 3.4.20

Case $(d = 3)$: $\zeta'_1 = \zeta_1 + 2a,\ \zeta'_2 = \zeta_2 + a\zeta_1 + a^2$.

Case $(d = 4)$: $\zeta'_1 = \zeta_1 + 3a,\ \zeta'_2 = \zeta_2 + 2a\zeta_1 + 3a^2,\ \zeta'_3 = \zeta_3 + a\zeta_2 + a^2\zeta_1 + a^3$.

Now we show the more explicit calculation in the proof of Theorem 3.4.19.

Let R be an integral domain with quotient field K and let $R[X]$ denote a polynomial ring over R. Let α be an element of an algebraic field extension of K, a an element of R, and $\varphi_{\alpha-a}(X)$ the monic minimal polynomial of $\alpha - a$ over K. Then the degree of $\varphi_{\alpha-a}(X)$ is unchanged for any $a \in R$. So let d be this constant value and write

$$\varphi_{\alpha-a}(X) := X^d + \eta_1^{(a)} X^{d-1} + \cdots + \eta_d^{(a)}$$

where $\eta_1^{(a)}, \ldots, \eta_d^{(a)} \in K$.

Let $\zeta_i^{(a)} := (\alpha - a)^i + \eta_1^{(a)}(\alpha - a)^{i-1} + \eta_2^{(a)}(\alpha - a)^{i-2} + \cdots + \eta_i^{(a)},\ \eta_i = \eta_i^{(0)}$, and $\zeta_i = \zeta_i^{(0)}$, for $i = 1, \ldots, d - 1$.

Let $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism defined by $\pi(X) = \alpha$. Define

$$I_{[\alpha]} := \bigcap_{i=1}^{d} (R :_R \eta_i)$$

where $(R :_R \eta_i) := \{c \in R | c\eta_i \in R\}$. Then $I_{[\alpha]}$ is an ideal of R. An element α is called an *anti-integral* element of degree d over R if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$.

Let $f : \mathbf{Z} \to R$ be a ring homomorphism sending $n \in \mathbf{Z}$ to $n \cdot 1 \in R$, where $\mathbf{Z}$ denotes the ring of integers and $F := \mathbf{Z}/\mathrm{Ker}(f)$, which is a subdomain of R.

Lemma 3.4.21

(1) $\varphi_{\alpha-a}(X) = \varphi_\alpha(X + a)$.
(2) $K(\zeta_1, \ldots, \zeta_{d-1}) = K(\zeta_1^{(a)}, \ldots, \zeta_{d-1}^{(a)}) = K(\alpha)$.

Proof (1) For some $\lambda_1, \ldots, \lambda_d \in K$, we can write

$$\varphi_\alpha(X) = (X - a)^d + \lambda_1(X - a)^{d-1} + \cdots + \lambda_d$$

since $K[X] = K[X-a]$. Then $\varphi_{\alpha-a}(X) = X^d + \lambda_1 X^{d-1} + \cdots + \lambda_d = \varphi_\alpha(X+a)$.
(2) It is clear that $K(\zeta_1^{(a)}) = K(\alpha)$. ∎

Lemma 3.4.22

$$\begin{aligned} \eta_k^{(a)} &= \binom{d}{k} a^k + \eta_1 \binom{d-1}{k-1} a^{k-1} + \eta_2 \binom{d-2}{k-2} a^{k-2} \\ &\quad + \cdots + \eta_{k-1} \binom{d-k+1}{1} a + \eta_k \end{aligned}$$

for $k = 1, \ldots, d$.

Proof Compare the coefficients of $\varphi_\alpha(X+a) = \varphi_{\alpha-a}(X)$ (*cf.* Lemma 3.4.1(1)). ∎

Lemma 3.4.23 *For* $k = 1, \ldots, d-1$,

$$\begin{aligned} \zeta_{k+1} &= \alpha\zeta_k + \eta_{k+1}, \quad \zeta_1 = \alpha + \eta_1 \\ \zeta_{k+1}^{(a)} &= (\alpha - a)\zeta_k^{(a)} + \eta_{k+1}^{(a)}, \ \textit{and}\ \zeta_1^{(a)} = (\alpha - a) + \eta_1^{(a)} \end{aligned}$$

Proof Trivial. $\square$

It is well-known that

$$\binom{n+i}{i+1} = \binom{n+i-1}{i+1} + \binom{n+i-1}{i}$$

for $i \geq 0$.

The aim is to give the following explicit equality.

Lemma 3.4.24

$$\zeta_k^{(a)} = \zeta_k + \binom{d-k}{1} a\zeta_{k-1} + \binom{d-k+1}{2} a^2\zeta_{k-2} + \binom{d-k+2}{3} a^3\zeta_{k-3}$$
$$+ \cdots + \binom{d-3}{k-2} a^{k-2}\zeta_2 + \binom{d-2}{k-1} a^{k-1}\zeta_1 + \binom{d-1}{k} a^k$$

i. e.,

$$\zeta_k^{(a)} = \sum_{i=0}^{k-1} \binom{d-k+i-1}{i} a^i \zeta_{k-i} + \binom{d-1}{k} a^k$$

for $k = 1, \ldots, d-1$.

Proof Let A_k denote the right-hand side of the above equality, i. e., $A_k := \zeta_k + \binom{d-k}{1} a\zeta_{k-1} + \binom{d-k+1}{2} a^2\zeta_{k-2} + \binom{d-k+2}{3} a^3\zeta_{k-3} + \cdots + \binom{d-3}{k-2} a^{k-2}\zeta_2 + \binom{d-2}{k-1} a^{k-1}\zeta_1 + \binom{d-1}{k} a^k$. We shall prove $\zeta_k^{(a)} = A_k$, by induction on k. For $k = 1$, $A_1 = \zeta_1 + \binom{d-1}{1} a = \zeta_1 + (d-1)a$, and $\zeta_1^{(a)} = (\alpha - a) + \eta_1^{(a)} = (\alpha - a) + \{\binom{d}{1} a + \eta_1\} = \alpha - a + da + \eta_1 = \alpha - a + da + (\zeta_1 - \alpha) = \zeta_1 + (d-1)a$. Hence $\zeta_1^{(a)} = A_1$. Assume that $\zeta_k^{(a)} = A_k$. We shall show that $\zeta_{k+1}^{(a)} = A_{k+1}$. By use of Lemmas 3.4.22 and 3.4.23, and by induction hypothesis, we have the following equality:

$$\begin{aligned}
& \zeta_{k+1}^{(a)} \\
= \; & (\alpha - a)\zeta_k^{(a)} + \eta_{k+1}^{(a)} \\
= \; & (\alpha - a)\left\{\zeta_k + \binom{d-k}{1} a\zeta_{k-1} + \binom{d-k+1}{2} a^2\zeta_{k-2} + \cdots \right. \\
& \left. + \binom{d-2}{k-1} a^{k-1}\zeta_1 + \binom{d-1}{k} a^k \right\} + \eta_{k+1}^{(a)}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\zeta_k + \binom{d-k}{1} a\alpha\zeta_{k-1} + \binom{d-k+1}{2} a^2\alpha\zeta_{k-2} + \cdots \\
&\quad + \binom{d-2}{k-1} a^{k-1}\alpha\zeta_1 + \binom{d-1}{k} a^k\alpha - a\zeta_k - \binom{d-k}{1} a^2\zeta_{k-1} \\
&\quad - \binom{d-k+1}{2} a^3\zeta_{k-2} - \cdots - \binom{d-2}{k-1} a^k\zeta_1 - \binom{d-1}{k} a^{k+1} + \eta_{k+1}^{(a)} \\
&= (\zeta_{k+1} - \eta_{k+1}) + \binom{d-k}{1} a(\zeta_k - \eta_k) + \binom{d-k+1}{2} a^2(\zeta_{k-1} - \eta_{k-1}) \\
&\quad + \cdots + \binom{d-2}{k-1} a^{k-1}(\zeta_2 - \eta_2) + \binom{d-1}{k} a^k(\zeta_1 - \eta_1) - a\zeta_k \\
&\quad - \binom{d-k}{1} a^2\zeta_{k-1} - \binom{d-k+1}{2} a^3\zeta_{k-2} - \cdots - \binom{d-2}{k-1} a^k\zeta_1 \\
&\quad - \binom{d-1}{k} a^{k+1} + \eta_{k+1}^{(a)} \\
&= \zeta_{k+1} + \binom{d-k}{1} a\zeta_k + \binom{d-k+1}{2} a^2\zeta_{k-1} + \cdots + \binom{d-2}{k-1} a^{k-1}\zeta_2 \\
&\quad + \binom{d-1}{k} a^k\zeta_1 - a\zeta_k - \binom{d-k}{1} a^2\zeta_{k-1} - \cdots - \binom{d-3}{k-2} a^{k-1}\zeta_2 \\
&\quad - \binom{d-2}{k-1} a^k\zeta_1 - \binom{d-1}{k} a^{k+1} - \binom{d-1}{k} a^k\eta_1 - \binom{d-2}{k-1} a^{k-1}\eta_2 \\
&\quad - \cdots - \binom{d-k}{1} a\eta_k - \eta_{k+1} + \left\{ \binom{d}{k+1} a^{k+1} + \binom{d-1}{k} a^k\eta_1 + \cdots \right. \\
&\quad \left. + \binom{d-k+1}{2} a^2\eta_{k-1} + \binom{d-k}{1} a\eta_k + \eta_{k+1} \right\} \\
&= \zeta_{k+1} + \left\{ \binom{d-k}{1} - 1 \right\} a\zeta_k + \left\{ \binom{d-k+1}{2} - \binom{d-k}{1} \right\} a^2\zeta_{k-1} \\
&\quad + \cdots + \left\{ \binom{d-1}{k} - \binom{d-2}{k-1} \right\} a^k\zeta_1 + \left\{ \binom{d}{k+1} \right. \\
&\quad \left. - \binom{d-1}{k} \right\} a^{k+1}
\end{aligned}
$$

$$\begin{aligned} &= \zeta_{k+1} + \binom{d-k-1}{1} a\zeta_k + \binom{d-k}{2} a^2\zeta_{k-1} + \cdots + \binom{d-2}{k} a^k\zeta_1 \\ &\quad + \binom{d-1}{k+1} a^{k+1} \\ &= A_{k+1} \end{aligned}$$

as desired. ∎

Remark 3.4.25 Lemma 3.4.24 shows that for $k = 1, \ldots, d-1$,

$$\zeta_k^{(a)} - \zeta_k \in F[a]\zeta_{k-1} + \cdots + F[a]\zeta_1 + F[a]$$ ∎

Corollary 3.4.26 *As R-submodules of $K[\alpha]$, $R + R\zeta_1 + \cdots + R\zeta_{d-1} = R + R\zeta_1^{(a)} + \cdots + R\zeta_{d-1}^{(a)}$, for any $a \in R$.*

Let H be an ideal of R and let $a \in R$. Put

$$C_H^{(a)} := R + H\zeta_1^{(a)} + \cdots + H\zeta_{d-1}^{(a)}$$

and

$$C_H := R + H\zeta_1 + \cdots + H\zeta_{d-1}$$

which are R-submodules of $K[\alpha]$.

Corollary 3.4.27 *$C_H = C_H^{(a)}$, for any $a \in R$.*

Proof By Corollary 3.4.26, we have $R + R\zeta_1 + \cdots + R\zeta_{d-1} = R + R\zeta_1^{(a)} + \cdots + R\zeta_{d-1}^{(a)}$, and hence $H + H\zeta_1 + \cdots + H\zeta_{d-1} = H + H\zeta_1^{(a)} + \cdots + H\zeta_{d-1}^{(a)}$. Thus $C_H = R + H\zeta_1 + \cdots + H\zeta_{d-1} = R + H\zeta_1^{(a)} + \cdots + H\zeta_{d-1}^{(a)} = C_H^{(a)}$. ∎

We close this section by showing:

Let R be an integral domain and α an anti-integral element over R. Let $\beta := (c\alpha - k)/(a\alpha - h)$ with $a, h, c, k \in R$ and $ak - hc \in R^\times$. We prove that β is anti-integral over R and that the equality $R[\beta] \cap R[\beta^{-1}] = R[\alpha] \cap R[\alpha^{-1}]$ holds.

Theorem 3.4.28 *Assume that α is an anti-integral element of degree d over R. Let $\beta = (c\alpha - k)/(a\alpha - h)$ with $a, h, c, k \in R$ and $ak - hc \in R^\times$. Then β is anti-integral over R and the equality $R[\beta] \cap R[\beta^{-1}] = R[\alpha] \cap R[\alpha^{-1}]$ holds.*

Note that when $\alpha, a, h, c, k \in \mathbf{C}$, the field of complex numbers, β is called a linear fractional transformation of α.

To begin with, we show the following lemma.

Lemma 3.4.29 *Assume that α is an anti-integral element of degree d over R. Then the following assertions hold:*

(1) *if $u \in R^{\times}$, then $u\alpha$ is an anti-integral element over R;*

(2) *if $a \in R$, then $\alpha - a$ is an anti-integral element over R;*

(3) *α^{-1} is an anti-integral element over R.*

Proof (2) and (3) are seen in Lemma 3.4.16.
(1) We have only to prove ($\Rightarrow$). Let $\pi_{u\alpha} : R[X] \to R[u\alpha]$ be a R-algebra homomorphism sending X to $u\alpha$ and $\varphi_{u\alpha}(X) \in K[X]$ be the monic minimal polynomial of $u\alpha$ over K. It is easy to see that $\deg(\varphi_{u\alpha}(X)) = \deg(\varphi_{\alpha}(X)) = d$. Since $(u\alpha)^d + u\eta_1(u\alpha)^{d-1} + \cdots + u^d\eta_d = u^d\varphi_{\alpha}(\alpha) = 0$, we have

$$\varphi_{u\alpha}(X) = X^d + u\eta_1 X^{d-1} + \cdots + u^d\eta_d$$

Since u is a unit in R, we have $I_{[u\alpha]} = \bigcap_{i=1}^{d}(R :_R u^i\eta_i) = \bigcap_{i=1}^{d}(R :_R \eta_i) = I_{[\alpha]}$. Take $f(X) \in R[X]$ such that $f(u\alpha) = 0$ and put $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0$ with $a_i \in R$. Then $f(u\alpha) = a_nu^n\alpha^n + a_{n-1}u^{n-1}\alpha^{n-1} + \cdots + a_0$. Note that $n \geq d$ because d is the degree of minimal polynomial of $u\alpha$. Set $g(X) := a_nu^nX^n + a_{n-1}u^{n-1}X^{n-1} + \cdots + a_0 \in R[X]$. Then $g(\alpha) = f(u\alpha) = 0$. So we have $g(X) \in \mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_{\alpha}(X)R[X]$ because α is anti-integral over R. Thus $g(X) = \varphi_{\alpha}(X)h(X)$ for some $h(X) \in I_{[\alpha]}R[X]$. Put $h(X) := b_0 + b_1X + \cdots + b_{n-d}X^{n-d}$ with $b_j \in I_{[\alpha]}$ and let $m(X) := b_0u^{-d} + b_1u^{-d-1}X + \cdots + b_{n-d}X^{n-d} \in R[X]$. Then $m(X) \subseteq I_{[\alpha]}R[X] = I_{[u\alpha]}R[X]$. Then it is easy to see that $g(X) = \varphi_{u\alpha}(uX)m(uX)$. Put $Y := uX$. Then $R[X] = R[Y]$ and $I_{[\alpha]}R[X] = I_{[u\alpha]}R[X] = I_{[u\alpha]}R[Y]$. We see that $f(Y) = g(X) = \varphi_{u\alpha}(Y)m(Y) \in R[Y]$. Therefore $f(X) = \varphi_{u\alpha}(X)m(X) \in R[X]$ by replacing Y by X formally. Hence $\mathrm{Ker}(\pi_{u\alpha}) \subseteq I_{[u\alpha]}\varphi_{u\alpha}(X)R[X]$ and the reverse inclusion is obvious. So we have $\mathrm{Ker}(\pi_{u\alpha}) = I_{[u\alpha]}\varphi_{u\alpha}(X)R[X]$, which means that $u\alpha$ is an anti-integral element of degree d over R. $\square$

Proof of Theorem 3.4.28. Considering Lemma 3.4.29 and Theorem 2.2.8, we have only to show that for each maximal ideal m of R, β or β^{-1} is anti-integral over R_m and $R_m[\beta] \cap R_m[\beta^{-1}] = R_m[\alpha] \cap R_m[\alpha^{-1}]$. Let m be a maximal ideal of R. Since $ak - hc \in R^{\times} \subseteq (R_m)^{\times}$, either (i) $ak \in (R_m)^{\times}$ or (ii) $hc \in (R_m)^{\times}$ holds.

Case (i): $ak \in (R_m)^\times$ means that both a and k are units in R_m. In this case, we have

$$\beta = \frac{c\alpha - k}{a\alpha - h} = \frac{\frac{c}{a}\alpha - \frac{k}{a}}{\alpha - \frac{h}{a}} = \frac{c}{a} + \frac{\frac{1}{a^2}(hc - ak)}{\alpha - \frac{h}{c}}$$

Put $e := h/a$, $f := c/a$, and $g := (1/a^2)(hc - ak)$. Then $e, f \in R_m$ and $g \in (R_m)^\times$; $\beta = f + g/(\alpha - e)$. Put $\gamma := g/(\alpha - e)$. Then $\beta = \gamma + f$. Since α is anti-integral over R_m, so is $\alpha - e$; so is $(\alpha - e)^{-1}$; so is $\gamma = g/(\alpha - e)$; and so is $\beta = \gamma + f$ by Lemma 3.4.28. Since γ is anti-integral over R_m, we have $R_m[\gamma] \cap R_m[\gamma^{-1}] = R_m[\gamma + f] \cap R_m[(\gamma + f)^{-1}] = R_m[\beta] \cap R_m[\beta^{-1}]$ by Theorem 2.2.8. Moreover $R_m[\gamma] \cap R_m[\gamma^{-1}] = R_m[g/(\alpha - e)] \cap R_m[(\alpha - e)/g] = R_m[1/(\alpha - e)] \cap R_m[\alpha - e] = R_m[1/(\alpha)] \cap R_m[\alpha]$, where the last equality follows from Theorem 2.2.8. Hence we have shown that β is anti-integral over R_m and that $R_m[\beta] \cap R_m[\beta^{-1}] = R_m[\alpha] \cap R_m[\alpha^{-1}]$ holds.

Case (ii): We have only to consider β^{-1} instead of β in the argument of Case (i). $\square$

Chapter 4

Denominator Ideals and Excellent Elements

Let R be a Noetherian domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic field extension L of the quotient field K of R and let $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$$

Then $\eta_i \in K$ $(1 \leq i \leq d)$ are uniquely determined by α. Let $I_{\eta_i} := R :_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$, the latter of which is called a *generalized denominator ideal* of α. We say that α is an *anti-integral element* if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. For $f(X) \in R[X]$, let $\mathsf{c}(f(X))$ denote the ideal of R generated by the coefficients of $f(X)$. For an ideal J of $R[X]$, let $\mathsf{c}(J)$ denote the ideal generated by the coefficients of the elements in J. If α is an anti-integral element, then $\mathsf{c}(\mathrm{Ker}(\pi)) = \mathsf{c}(I_{[\alpha]}\varphi_\alpha(X)R[X]) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$.

Put $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Let $\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) | \mathrm{depth}(R_p) = 1\}$, then α is called a *super-primitive* element. It is known that a super-primitive element is an anti-integral element (*cf.* Theorem 2.2.8). It is known that any algebraic element over a Krull domain R is anti-integral over R (*cf.* Theorem 2.2.9). When α is an element in K, $\varphi_\alpha(X) = X - \alpha$. So we have $J_{[\alpha]} = I_{[\alpha]}(1, \alpha) = I_\alpha(1, \alpha) = I_\alpha + \alpha I_\alpha = I_\alpha + I_{\alpha^{-1}}$, where $I_\alpha := R :_R \alpha$, a *denominator ideal* of $\alpha \in K$.

4.1 Denominator Ideals and Flatness (I)

For the flatness of anti-integral extension, denominator ideals play important roles. In fact we have the following proposition.

Theorem 4.1.1 *Assume that $\alpha \in L$ is an anti-integral element of degree d over R. If $I_{[\alpha]}R[\alpha] = R[\alpha]$, then $R[\alpha]$ is flat over R.*

Proof Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, there exist elements $a_i \in I_{[\alpha]}$ $(0 \leq i \leq n)$ such that $1 = \sum_{i=0}^{n} a_i\alpha^i$. Take $p \in \mathrm{Spec}(R)$. Replacing R and $R[\alpha]$ by R_p and $R[\alpha]_p$, we may assume that R is a local ring with maximal ideal m. If $I_{[\alpha]}$ is not contained in m, then $I_{[\alpha]} = R$, and hence $\varphi_\alpha(X) \in R[X]$. Thus $R[\alpha]$ is a free R-module in this case. If $I_{[\alpha]} \subseteq m$, then we put $f(X) = 1 - \sum_{i=0}^{n} a_i X^i$. Then $f(\alpha) = 0$. Since α is an anti-integral element over R, there exist elements $g_i(X), h_i(X) \in R[X]$ $(0 \leq i \leq n)$ such that $f(X) = \sum_{i=0}^{n} g_i(X)h_i(X)$ and $g_i(X) \in I_{[\alpha]}\varphi_\alpha(X)$ for all i. Hence we have $f(X) \in m[X]$. However, considering a constant term of $f(X)$, we see that $1 - a_0 \in m$. Since $a_0 \in I_{[\alpha]} \subseteq m$, this is a contradiction. Therefore we see $J_{[\alpha]} = \mathrm{c}(I_{[\alpha]}\varphi_\alpha(X)) = R$. Hence $R[\alpha]$ is flat over R by Theorem 2.3.6. ∎

Next we determine the nonflat locus for an integral extension. For this purpose, we prove the following lemma.

Lemma 4.1.2 *Assume that $R[\alpha]$ is a free R-module of rank d. Then we have $R[\alpha] = R + R\alpha + \cdots + R\alpha^{d-1}$.*

Proof Put $B := R + R\alpha + \cdots + R\alpha^{d-1} \subseteq R[\alpha]$. Then we have only to show that $R[\alpha]_p = B_p$ for all $p \in \mathrm{Spec}(R)$. Hence we may assume that R is a local ring with maximal ideal m. Since $R[\alpha]$ is free R-module of rank d, $R[\alpha]/mR[\alpha]$ is a free R/m-module of rank d, and hence $R[\alpha]/mR[\alpha]$ is a d-dimensional vector space over R/m. Note that $R[\alpha]/mR[\alpha] = (R/m)[\overline{\alpha}]$ with $\overline{\alpha} \in R[\alpha]/mR[\alpha]$. Now suppose that $\overline{1}, \overline{\alpha}, \ldots, \overline{\alpha}^s$ $(s < d)$ are linearly dependent over R/m and $\overline{1}, \overline{\alpha}, \ldots, \overline{\alpha}^{s-1}$ are linearly independent over R/m. Then we have $R[\alpha]/mR[\alpha] = R/m + (R/m)\overline{\alpha} + \cdots + (R/m)\overline{\alpha}^{s-1}$. This contradicts the fact that $\dim(R[\alpha]/mR[\alpha]) = d$. Hence $\overline{1}, \overline{\alpha}, \ldots, \overline{\alpha}^{d-1}$ are linearly independent over R/m and we see that $\overline{1}, \overline{\alpha}, \ldots, \overline{\alpha}^d$ are linearly dependent over R/m. Therefore we see that $R[\alpha] = R + R\alpha + \cdots + R\alpha^{d-1}$ by Nakayama's lemma, as desired. ∎

Hence we have the following:

Proposition 4.1.3 *Assume that $R[\alpha]$ is integral over R. Then $V(I_{[\alpha]}) = \{p \in \mathrm{Spec}(R) \mid R[\alpha]_p$ is not flat over $R\}$.*

Proof Let $[K(\alpha) : K] = d$ and consider $p \in \mathrm{Spec}(R)$ such that $I_{[\alpha]}$ is not contained in p. Then $\varphi_\alpha(X)$ is monic polynomial of degree d over R_p. Since $[K(\alpha) : K] = d$, we see that $R[\alpha]_p = R_p[\alpha]$ is a free R_p-module of rank d, and $R[\alpha]_p$ is a flat R_p-module, then $R[\alpha]_p = R_p + R_p\alpha + \cdots + R_p\alpha^{d-1}$ by Lemma 4.1.2. Thus there exists a monic relation of α of degree d over R_p.

Therefore $\varphi_\alpha(X) \in R_p[X]$, and $I_{[\alpha]}$ is not contained in p. This completes the proof. $\square$

Proposition 4.1.4 *Assume that* $I_{[\alpha]}R[\alpha] = R[\alpha]$. *Then* $V(I_{[\alpha]} + I_{[\alpha^{-1}]}) \supseteq \{p \in \mathrm{Spec}(R) | R[\alpha]_p$ *is not flat over* $R_p\}$.

Proof Let $[K(\alpha) : K] = d$ and assume that p does not contain $I_{[\alpha]} + I_{[\alpha^{-1}]}$ for $p \in \mathrm{Spec}(R)$. Then p does not contain $I_{[\alpha]}$ or $I_{[\alpha^{-1}]}$. First, assume that p does not contain $I_{[\alpha]}$. Then $\varphi_\alpha(X) \in R_p[X]$. Hence $R[\alpha]_p$ is a free R_p-module. Next, assume that p does not contain $I_{[\alpha^{-1}]}$ and $p \supseteq I_{[\alpha]}$. Since p does not contain $I_{[\alpha^{-1}]}$, by the definition of $I_{[\alpha^{-1}]}$, we have that $R_p[\alpha^{-1}]$ is a free R_p-module of rank d. Since $I_{[\alpha]}R[\alpha] = R[\alpha]$ by the assumption, it holds that α^{-1} is integral over R_p. Thus we have an integral relation $(\alpha^{-1})^n + (\alpha^{-1})^{n-1} + \cdots + a_n = 0$, $a_i \in R_p$ $(1 \le i \le n)$, and hence $\alpha^{-1} = -(a_1 + a_2\alpha + \cdots + a_n\alpha^{n-1}) \in R_p[\alpha]$. Therefore $R[\alpha]_p = R_p[\alpha] = R_p[\alpha, \alpha^{-1}]$. Since $R[\alpha]_p$ is flat over $R_p[\alpha^{-1}]$, we conclude that $R[\alpha]_p$ is flat over R_p. $\square$

Remark 4.1.5 Under the above circumstances, we can see easily that if $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$ and $I_{[\alpha]}R[\alpha] = R[\alpha]$, then $R[\alpha]$ is flat over R. $\square$

4.2 Excellent Elements of Anti-Integral Extensions

Now we are interested in giving a characterization for an anti-integral extension $R[\alpha]$ of R to be flat.

Let $R[\alpha]$ be an anti-integral extension of R and let γ be an element of $R[\alpha]$. Then we consider the following conditions:

(1) $R[\alpha]$ is flat over R;

(2) $R[\alpha, \gamma^{-1}]$ is flat over R.

It is easy to see that (1) implies (2). Our question is what γ implies (2) $\Rightarrow$ (1). For this, we introduce the following definition:

An element γ in $R[\alpha]$ is said to be an *excellent* element in $R[\alpha]$ if there exist elements $c_0, c_1, \ldots, c_n \in R$ such that $\gamma = c_0 + c_1\alpha + \cdots + c_n\alpha^n$ and $(c_0, c_1, \ldots, c_n)R = R$.

It is obvious that α itself is an excellent element in $R[\alpha]$. Our main theorem in this section is the following:

Let $R[\alpha]$ be an anti-integral extension of R and let $\gamma \in R[\alpha]$ be an excellent element. Then the following statements are equivalent:

(i) $R[\alpha]$ is flat over R;

(ii) $R[\alpha, \gamma^{-1}]$ is flat over R.

We now start with the following proposition.

Proposition 4.2.1 *Assume that α is anti-integral over R and let $\gamma \in R[\alpha]$ be a nonzero element. Then the following statements are equivalent:*

(1) *$R[\alpha, \gamma^{-1}]$ is flat over R;*

(2) $\gamma \in \sqrt{J_{[\alpha]}R[\alpha]}$.

Proof Put $A = R[\alpha, \gamma^{-1}]$.
(2) $\Rightarrow$ (1): Take $\overline{P} \in \mathrm{Spec}(A)$, and let $P = \overline{P} \cap R[\alpha]$ and $P \cap R = p$. It is obvious that $\gamma \notin P$. Suppose that $p \supseteq J_{[\alpha]}$. Then $P \supseteq \sqrt{J_{[\alpha]}R[\alpha]} \ni \gamma$, which is a contradiction. Hence we have $p \not\supseteq J_{[\alpha]}$. Since $R_p[\alpha]$ is flat over R_p (Corollary 2.2.6) and $R[\alpha]_P$ is flat over $R_p[\alpha]$, we see that $R[\alpha]_P$ is flat over R_p. Note that $R[\alpha]_P = A_P$ because $\gamma \notin P$. Thus $A_{\overline{P}}$ is flat over R_p.
(1) $\Rightarrow$ (2): Let P be a minimal prime divisor of $\sqrt{J_{[\alpha]}R[\alpha]}$. We have only to show that $\gamma \in P$. Suppose the contrary, that is, suppose that $\gamma \notin P$. Put $p = P \cap R$. Then $J_{[\alpha]} \subseteq p$. Hence p is a blowing-up point (Theorem 2.2.5), that is, $R_p[\alpha]/pR_p[\alpha]$ is isomorphic to a polynomial ring $R_p/pR_p[T] = k(p)[T]$. Thus $pR_p[\alpha]$ is a prime ideal of $R_p[\alpha]$. Since $PR_p[\alpha] \supseteq pR_p[\alpha] \supseteq \sqrt{J_{[\alpha]}R[\alpha]}$ and the minimality of P, we have $PR_p[\alpha] = pR_p[\alpha]$. We also have

$$(*) \qquad \mathrm{Tr.deg}_{k(p)} k(P) = 1$$

Since $\gamma \notin P$, it follows that $A_P = R[\alpha, \gamma^{-1}]_P = R[\alpha]_P$. The flatness of A over R implies that $R[\alpha]_P = A_P$ is flat over R_p. Hence by [M1,(13.B)], we have

$$\begin{aligned}
\dim A_P &= \dim R_p + \dim A_P/pA_P \\
&= \dim R_p + \dim R[\alpha]_P/pR[\alpha]_P \\
&= \dim R_p + \dim R[\alpha]_P/PR[\alpha]_P \\
&= \dim R_p + \dim k(P) \\
&= \dim R_p
\end{aligned}$$

Thus we have

$$(**) \qquad \dim A_P = \dim R_p$$

On the other hand, we have $\mathrm{Tr.deg}_{R_p} R_p[\alpha] = 0$ since α is algebraic over R. Note that $R_p[\alpha]$ is an R_p-algebra of finite type. So from [M1, (14.C) Theorem 23],

it follows that

$$\begin{aligned}
\mathrm{ht}(P) &= \mathrm{ht}(PR_p) \\
&\leq \mathrm{ht}(pR_p) + \mathrm{Tr.deg}_{R_p} R_p[\alpha] - \mathrm{Tr.deg}_{k(p)} k(P) \\
&= \mathrm{ht}(p) - \mathrm{Tr.deg}_{k(p)} k(P) \\
&= \mathrm{ht}(p) - 1 \ \ (by\ (*))
\end{aligned}$$

This implies $\mathrm{ht}(P) \leq \mathrm{ht}(p) - 1$, which contradicts the equality $(**)$. ∎

Remark 4.2.2 Let $f(X)$ and $g(X)$ be elements in a polynomial ring $R[X]$. If $\mathsf{c}(f(X)) = \mathsf{c}(g(X)) = R$, then $\mathsf{c}(f(X)g(X)) = R$. Indeed, we may assume that R is a local domain (R, M). Let $f(X) = a_0X^s + \cdots + a_s$ and $g(X) = b_0X^t + \cdots + b_t$, where $a_i \in M$ (respectively $b_j \in M$) for $0 \leq i \leq m-1$ (respectively for $0 \leq j \leq n-1$) and $a_n \notin M$ (respectively $b_m \notin M$). Then the coefficient of X^{n+m} in $f(X)g(X)$ is not in M. ∎

An excellent element has the following property, that is, the set of excellent elements are closed under multiplication.

Proposition 4.2.3 *If $\beta, \gamma \in R[\alpha]$ are excellent elements, then $\beta\gamma$ is also an excellent element.*

Proof Since β, γ are excellent elements, we can write $\beta = a_0 + a_1\alpha + \cdots + a_n\alpha^n$ and $\gamma = b_0 + b_1\alpha + \cdots + b_t\alpha^t$ such that $(a_0, \ldots, a_n)R = (b_0, \ldots, b_t)R = R$ for some $a_i, b_j \in R$. Put $f(X) = a_0 + a_1X + \cdots + a_nX^n$ and $g(X) = b_0 + b_1X + \cdots + b_tX^t$. Then $\mathsf{c}(f(X)) = \mathsf{c}(g(X)) = R$. Thus by Remark above, $\mathsf{c}(f(X)g(X)) = R$, which means that $\beta\gamma$ is an excellent element in $R[\alpha]$. ∎

In view of Proposition 4.2.3, we have the following result.

Proposition 4.2.4 *Assume that α is an anti-integral element over R and that $\gamma \in R[\alpha]$ is an excellent element. Then the following statements are equivalent:*

(1) $\gamma \in \sqrt{J_{[\alpha]}R[\alpha]}$;

(2) $J_{[\alpha]} = R$.

Proof The implication (2) $\Rightarrow$ (1) is trivial.
(1) $\Rightarrow$ (2): Since $\gamma \in \sqrt{J_{[\alpha]}R[\alpha]}$, we have $\gamma^n \in J_{[\alpha]}R[\alpha]$ for some $n > 0$. Since γ is an excellent element in $R[\alpha]$, it follows that γ^n is also an excellent

element by Proposition 4.2.3, and so there exist $c_i \in R$ such that $\gamma^n = c_0 + c_1\alpha + \cdots + c_t\alpha^t$ and $(c_0, c_1, \ldots, c_t)R = R$. On the other hand, since $\gamma^n \in J_{[\alpha]}R[\alpha]$, we see that $\gamma^n = a_0 + a_1\alpha + \cdots + a_m\alpha^m$ for some $a_i \in J_{[\alpha]}$. Put $f(X) = a_0 + a_1X + \cdots + a_mX^m - (c_0 + c_1X + \cdots + c_tX^t)$. Then $f(\alpha) = 0$ and hence $f(X) \in \operatorname{Ker}\pi = I_{[\alpha]}\varphi_\alpha R[X]$ (because α is anti-integral over R). Thus $\mathrm{c}(f(X)) \subseteq \mathrm{c}(\operatorname{Ker}\pi) = \mathrm{c}(I_{[\alpha]}\varphi_\alpha) = J_{[\alpha]}$. Since $a_0, \ldots, a_m$ belong to $J_{[\alpha]}$, we conclude that $c_0, \ldots, c_t \in J_{[\alpha]}$, whence $J_{[\alpha]} = R$. ∎

Combining the above result with Proposition 4.2.1, we have the following result.

Theorem 4.2.5 *Assume that α is an anti-integral element over R and let $\gamma \in R[\alpha]$ be an excellent element. Then the following statements are equivalent to each other:*

(1) *$R[\alpha, \gamma^{-1}]$ is flat over R;*

(2) *$R[\alpha]$ is flat over R.*

Proof (2) $\Rightarrow$ (1) is trivial.
(1) $\Rightarrow$ (2): Since $R[\alpha, \gamma^{-1}]$ is flat over R, it follows that $\gamma \in \sqrt{J_{[\alpha]}R[\alpha]}$ by Proposition 4.2.1. So we have $J_{[\alpha]} = R$ by Proposition 4.2.4, which means that $R[\alpha]$ is flat over R by Theorem 2.3.6. ∎

Noting that α itself is an excellent element in $R[\alpha]$, we have the following Corollary.

Corollary 4.2.6 *Assume that α is an anti-integral element over R. The following statements are equivalent:*

(1) *$R[\alpha]$ is flat over R;*

(2) *$R[\alpha, \alpha^{-1}]$ is flat over R;*

(3) *$\alpha \in \sqrt{J_{[\alpha]}R[\alpha]}$;*

(4) *$J_{[\alpha]} = R$;*

(5) *every excellent element belongs to $\sqrt{J_{[\alpha]}R[\alpha]}$.*

Proof (1) $\Leftrightarrow$ (2) follows from Theorem 4.2.5, (2) $\Leftrightarrow$ (3) follows from Proposition 4.2.1, and (3) $\Leftrightarrow$ (4) and (4) $\Leftrightarrow$ (5) follow from Proposition 4.2.4. ∎

Finally, we give a characterization of excellent elements.

Proposition 4.2.7 *Assume that α is an anti-integral element over R and let $\gamma \in R[\alpha]$. Then the following statements are equivalent:*

(1) *γ is an excellent element;*

(2) *$(a_0, a_1, \ldots, a_n)R + J_{[\alpha]} = R$ for each representation $\gamma = a_0 + a_1\alpha + \cdots + a_n\alpha^n$ of γ with $a_i \in R$.*

Proof (1) $\Rightarrow$ (2): Suppose that $(a_0, a_1, \ldots, a_n)R + J_{[\alpha]} \neq R$ for some representation $\gamma = a_0 + a_1\alpha + \cdots + a_n\alpha^n$. Then there exists $p \in \mathrm{Spec}(R)$ such that $(a_0, a_1, \ldots, a_n)R + J_{[\alpha]} \subseteq p$. Since γ is an excellent element in $R[\alpha]$, we have $\gamma = c_0 + c_1\alpha + \cdots + c_t\alpha^t$ with $(c_0, c_1, \ldots, c_t)R = R$ for some $c_i \in R$. Put $f(X) = (c_0 + c_1X + \cdots + c_tX^t) - (a_0 + a_1X + \cdots + a_nX^n)$. Then we see $f(\alpha) = 0$, and hence $\mathrm{c}(f(X)) \subseteq J_{[\alpha]} \subseteq p$. Since $a_0, a_1, \ldots, a_n \in p$, it follows that $c_0, c_1, \ldots, c_t \in p$. This implies that $p = R$, a contradiction.
(2) $\Rightarrow$ (1): Since $(a_0, a_1, \ldots, a_n)R + J_{[\alpha]} = R$ for any representation $\gamma = a_0 + a_1\alpha + \cdots + a_n\alpha^n$ of γ with $a_i \in R$, we have $1 = (\sum a_ix_i) + y$ for some $x_i \in R$, $y \in J_{[\alpha]}$. Since $J_{[\alpha]} = \mathrm{c}(I_{[\alpha]}\varphi_\alpha(X))$ by definition, there exist $c_i \in J_{[\alpha]}$, $f_i(X) \in I_{[\alpha]}\varphi_\alpha(X)$, and $b_i \in R$ such that $y = \sum_{i=1}^t c_ib_i$, c_i is one of the coefficients of $f_i(X)$, and $f_i(\alpha) = 0$ for each i. Thus we have $\gamma = a_0 + a_1\alpha + \cdots + a_n\alpha^n = a_0 + a_1\alpha + \cdots + a_n\alpha^n + \sum_{i=1}^t b_i\alpha^{n+d_i}f_i(\alpha)$ for a sufficiently large integer $d_i > 0$ and $(a_0, a_1, \ldots, a_n, b_1c_1, b_2c_2, \ldots, b_tc_t) = R$. It follows that γ is an excellent element in $R[\alpha]$. $\square$

4.3 Flatness and LCM-Stableness

Let C be a ring extension of a ring D. We say that C is LCM-*stable* over D if $(aD \cap bD)C = aC \cap bC$ for any $a, b \in D$.

Let R be a Noetherian integral domain and α be a nonzero element of an algebraic field extension of the quotient field K of R. Let $R\langle\alpha\rangle := R[\alpha] \cap R[\alpha^{-1}]$, let $I_\alpha^{R\langle\alpha\rangle} := \{b \in R\langle\alpha\rangle | b\alpha \in R\langle\alpha\rangle\}$, and $J_\alpha^{R\langle\alpha\rangle} := I_\alpha^{R\langle\alpha\rangle} + \alpha I_\alpha^{R\langle\alpha\rangle}$. In the birational case, F. Richman, H. Uda, J. Sato, and K. Yoshida showed that if $R[\alpha]$ is LCM-stable over R, then $R[\alpha]$ is flat over R ([R], [SY2], and [U]). But, in general, this is not the case [U] (*cf.* Example 4.3.14)].

Lemma 4.3.1 *Assume that α is an anti-integral element over R. Then $R[\alpha]$ is flat over R if and only if $R[\alpha]$ is flat over $R\langle\alpha\rangle$.*

Proof Since $R\langle\alpha\rangle[\alpha] \cap R\langle\alpha\rangle[\alpha^{-1}] \subseteq R\langle\alpha\rangle = R\langle\alpha\rangle$, it follows that $R\langle\alpha\rangle[\alpha] \cap R\langle\alpha\rangle[\alpha^{-1}] = R\langle\alpha\rangle$, and hence α is an anti-integral element over $R\langle\alpha\rangle$ (*cf.* Definition 1.1.4).

($\Leftarrow$) Suppose that $R[\alpha]$ is flat over $R\langle\alpha\rangle$. Since $R\langle\alpha\rangle$ is integral over R, $R\langle\alpha\rangle$ is not a blowing-up at any point in $\mathrm{Spec}(R)$. Then $R[\alpha]$ is flat over R by Theorem 2.3.6.
($\Rightarrow$) Suppose that $R[\alpha]$ is flat over R. Take $P \in \mathrm{Spec}(R[\alpha])$ and put $p = P \cap R$. Since $k(P)/k(p)$ is an algebraic field extension, $R[\alpha]$ is not a blowing-up at any point in $\mathrm{Spec}(R)$. Then Theorem 2.3.6 implies that $R[\alpha]$ is flat over $R\langle\alpha\rangle$. $\square$

Lemma 4.3.2 *Let $a, b \in R$ be an R-regular sequence. Then there exists $P \in \mathrm{Dp}_1(R[b/a])$ such that $P \cap R$ is of* grade > 1.

Proof Put $\alpha = b/a$. Since a, b is an R-regular sequence, we have that $I_\alpha = aR$. Since $J_\alpha = (a, b)R$ is of grade > 1, α is super-primitive, and so α is anti-integral. Now $R[\alpha]/J_\alpha R[\alpha] \cong (R[X]/I_\alpha(X - \alpha)R[X])/J_\alpha R[X]) \cong R[X]/J_\alpha R[X] \cong (R/J_\alpha)[X]$. Thus, in particular, $aR[\alpha] \neq R[\alpha]$. Let P be a minimal prime divisor of $aR[\alpha]$. It then follows that $a, b \in p = P \cap R$. Hence $\mathrm{depth}(R_p) > 1$. $\square$

Using Lemma 4.3.2, we can give the following result.

Proposition 4.3.3 *If $\alpha \in K$ and I_α is an invertible ideal of R, then the following statements are equivalent:*

(1) *for each $P \in \mathrm{Dp}_1(R[\alpha])$, $P \cap R$ belongs to $\mathrm{Dp}_1(R)$;*
(2) *$R[\alpha]$ is flat over R.*

Proof We may assume that R is a local ring. Then $I_\alpha = aR$ for some $a \in R$ because I_α is invertible. So there exists an element $b \in R$ such that $\alpha = b/a$. Assume that (1) holds. Then by Lemma 4.3.2, a, b cannot be an R-regular sequence. Since $I_\alpha = aR$, we have $(a, b)R = R$. Now $x, y \in R$ satisfy $ax + by = 1$. Then $1/a = x + \alpha \in R[\alpha]$, and so $R[\alpha] = R[1/a]$, which implies that $R[\alpha]$ is flat over R. Conversely, assume that (2) holds. Let $P \in \mathrm{Dp}_1(R[\alpha])$, and put $p = P \cap R$. If $a, b \in p$ is an R-regular sequence, then $aR : bR = aR$. Since $R[\alpha]$ is flat over R, we conclude that $aR[\alpha] : bR[\alpha] = aR[\alpha]$. Thus a, b is an $R[\alpha]$-regular sequence, which is a contradiction. $\square$

Lemma 4.3.4 *Assume that α is an anti-integral element over R. Then we have that $\sqrt{J_\alpha^{R\langle\alpha\rangle}} \supseteq J_{[\alpha]}$.*

Proof Take $R \in \mathrm{Spec}(R\langle\alpha\rangle)$ and put $p = P \cap R$. Now suppose that $p \not\supseteq J_{[\alpha]}$. Since $R[\alpha]_p$ is flat over R_p (Theorem 2.3.6) and $R[\alpha]_p$ is anti-integral over

R_p, we see that $R[\alpha]_p$ is flat over $R\langle\alpha\rangle_p$ by Lemma 4.3.1. Hence $P \not\supseteq J_\alpha^{R\langle\alpha\rangle}$ (Theorem 2.3.6). Thus we have $\sqrt{J_\alpha^{R\langle\alpha\rangle}} \supseteq J_{[\alpha]}$. ∎

Proposition 4.3.5 *Assume that α is a super-primitive element over R. Then α is also a super-primitive element over $R\langle\alpha\rangle$.*

Proof Take $P \in \mathrm{Dp}_1(R\langle\alpha\rangle)$. Suppose that $P \supseteq J_\alpha^{R\langle\alpha\rangle}$. Then it follows from Lemma 4.3.4 that $p = P \cap R \supseteq J_{[\alpha]}$. So by Lemma 3.1.28, we see that $p \in \mathrm{Dp}_1(R)$. This contradicts that α is a super-primitive over R. Hence α is super-primitive over $R\langle\alpha\rangle$. ∎

Theorem 4.3.6 *Assume that α is a super-primitive element over R. Then the following statements are equivalent:*

(1) *$I_\alpha^{R\langle\alpha\rangle}$ is an invertible ideal of $R\langle\alpha\rangle$ and $P \cap R \in \mathrm{Dp}_1(R)$ for each $P \in \mathrm{Dp}_1(R[\alpha])$;*
(2) *$R[\alpha]$ is a flat R-module.*

Proof (2) $\Rightarrow$ (1) Suppose that $R[\alpha]$ is flat over R. By Lemma 4.3.1, $R[\alpha]$ is flat over $R\langle\alpha\rangle$, $R[\alpha] = R\langle\alpha\rangle[\alpha]$ is anti-integral over $R\langle\alpha\rangle$ by Proposition 4.3.5, it follows from Proposition 2.5.4 that $R\langle\alpha\rangle = J_\alpha^{R\langle\alpha\rangle} = I_\alpha^{R\langle\alpha\rangle}(1, \alpha)$. Hence $I_\alpha^{R\langle\alpha\rangle}$ is an invertible ideal of $R\langle\alpha\rangle$. Then by Proposition 4.3.3 and Lemma 3.1.28, we see that $P \cap R \in \mathrm{Dp}_1(R)$ for each $P \in \mathrm{Dp}_1(R[\alpha])$.
(1) $\Rightarrow$ (2) First, suppose that $I_\alpha^{R\langle\alpha\rangle}R[\alpha] = R[\alpha]$. Then $R[\alpha]$ is flat over $R\langle\alpha\rangle$ (Theorem 4.1.1 and Proposition 4.3.5). Next, suppose that $I_\alpha^{R\langle\alpha\rangle}R[\alpha] \neq R[\alpha]$. Let P be a prime divisor of $I_\alpha^{R\langle\alpha\rangle}R[\alpha]$. Put $q = P \cap R\langle\alpha\rangle$, and localize at q. We have that $I_\alpha^{R\langle\alpha\rangle}R[\alpha]_q$ is a principal ideal of $R[\alpha]_q$. Hence $P \in \mathrm{Dp}_1(R[\alpha])$. Further, we have $J_\alpha^{R\langle\alpha\rangle} = I_\alpha^{R\langle\alpha\rangle} + \alpha I_\alpha^{R\langle\alpha\rangle} \subseteq q$. Then Proposition 4.3.5 implies that $\mathrm{depth}(R\langle\alpha\rangle_q) > 1$. On the other hand, we see that Lemma 4.3.4 implies that $p = q \cap R \supseteq J_{[\alpha]}$. However, we have that $\mathrm{depth}(R_p) > 1$ since α is a super-primitive element over R, which is a contradiction. Therefore $I_\alpha^{R\langle\alpha\rangle}R[\alpha] = R[\alpha]$, and so $R[\alpha]$ is flat over $R\langle\alpha\rangle$. Hence by Lemma 4.3.1, $R[\alpha]$ is flat over R. ∎

Lemma 4.3.7 *If $R[\alpha]$ is LCM-stable over R, then $P \cap R \in \mathrm{Dp}_1(R)$ for each $P \in \mathrm{Dp}_1(R[\alpha])$.*

Proof Let $P \in \mathrm{Dp}_1(R[\alpha])$, and put $p = P \cap R$. Suppose that $\mathrm{depth}(R_p) > 1$. Then there exists an R_p-regular sequence a, b of R_p. Since $R[\alpha]$ is LCM-stable over R, we know that $(aR_p \cap bR_p)R[\alpha]_p = aR[\alpha]_p \cap bR[\alpha]_p = (abR_p)R[\alpha]_p$. Hence a, b is an $R[\alpha]_p$-regular sequence, a contradiction. ∎

Theorem 4.3.8 *Assume that α is an anti-integral element over R. If $I_\alpha^{R\langle\alpha\rangle}$ is an invertible ideal of $R\langle\alpha\rangle$ and $R[\alpha]$ is LCM-stable over R, then $R[\alpha]$ is flat over R.*

Proof First note that by Lemma 4.3.7, we have $P \cap R \in \mathrm{Dp}_1(R)$ for each $P \in \mathrm{Dp}_1(R[\alpha])$. Since $I_\alpha^{R\langle\alpha\rangle}$ is an invertible ideal of $R\langle\alpha\rangle$, α is a super-primitive element over $R\langle\alpha\rangle$ (Theorem 2.3.10). So we conclude that $R[\alpha]$ is flat over R by Theorem 4.3.6. ∎

Lemma 4.3.9 *Let $R \subseteq A$ be an extension of integral domains such that A is LCM-stable over R, and let P be a prime ideal of A of depth 1. Then $P \cap R$ is also of depth 1.*

Proof Put $p = P \cap R$ and assume that $\mathrm{depth}(R_p) > 1$. Then there exists an R_p-regular sequence a, b in pR_p. We have $aR_p \cap bR_p = abR_p$. Since A is LCM-stable over R, it follows that $aA_p \cap bA_p = abA_p$, and hence $aA_P \cap bA_P = abA_P$. This shows that a, b is an A_P-regular sequence in PA_P, which contradicts $\mathrm{depth}(A_P) = 1$. ∎

Lemma 4.3.10 *Let $A := R[X]/I$, where I is a nonzero prime ideal of $R[X]$ with $I \cap R = (0)$, and let p be a prime ideal of R containing $\mathsf{c}(I)$. Then pA is a prime ideal of A such that $pA \cap R = p$ and that $\mathrm{ht}(pA) \leq \mathrm{ht}(p) - 1$. In particular, $\mathrm{ht}(\mathsf{c}(I)) \geq 2$.*

Proof Since p contains $\mathsf{c}(I)$, we have $I \subseteq pR[X]$, and hence $pA = pR[X]/I$. Thus pA is a prime ideal satisfying $pA \cap R = p$. Recall that $\mathrm{ht}(pR[X]) = \mathrm{ht}(p)$ because R is Noetherian. Therefore

$$\mathrm{ht}(pA) = \mathrm{ht}(pR[X])/I) \leq \mathrm{ht}(pR[X]) - 1 = \mathrm{ht}(p) - 1$$

Note that $\mathrm{ht}(pA) \neq 0$, and if $\mathrm{ht}(pA) = 0$, then $pR[X] = I$, which contradicts $I \cap R = (0)$. Hence we have $\mathrm{ht}(p) \geq 2$, which implies that $\mathrm{ht}(\mathsf{c}(I)) \geq 2$. ∎

Lemma 4.3.11 *Let the notation and the assumption be the same as in the previous lemma, and assume that A is LCM-stable over R. If $\mathrm{ht}(p) = 2$, then $\mathrm{depth}(R_p) = 1$.*

Proof Let $P = pA$. Then we have $P \cap R = p$ and $\mathrm{ht}(P) = 1$ by Lemma 4.3.10. The assertion then follows from Lemma 4.3.9. ∎

Now we are in a position to show the following theorem.

Theorem 4.3.12 *Assume that α is a super-primitive element over R and that $J_{[\alpha]} \neq R$. If $R[\alpha]$ is LCM-stable over R, then* $\mathrm{ht}(J_{[\alpha]}) \geq 3$.

Proof Our conclusion is an immediate consequence of Lemma 4.3.11. ∎

Concerning Lemma 4.3.11, we note the following.

Proposition 4.3.13 *Assume that R satisfies the S_2-condition, that is, prime ideals of depth 1 are of height 1, and α is an anti-integral element over R. Then α is super-primitive over R.*

Proof Let p be a prime ideal of R containing $J_{[\alpha]}$. Then we have $\mathrm{ht}(p) \geq 2$ by Lemma 4.3.10, and hence $\mathrm{depth}(R_p) \geq 2$ because of the S_2-condition. This shows that $J_{[\alpha]}$ is not contained in any depth 1 prime ideal of R, and therefore α is super-primitive over R. ∎

We conclude this section by giving the following example, which shows that the converse statement of Theorem 4.3.12 does not hold in general.

Example 4.3.14 Let $B := k[s, t, u]$ be a polynomial ring in three variables over a field k and α be a root of the polynomial $sX^2 + tX + u$ in a suitable extension field L of $k(s, t, u)$. Let $R := B[s\alpha]$ and consider the extension $R \subseteq A := R[\alpha]$. We shall show that $\mathrm{ht}(J_{[\alpha]}) = \mathrm{grade}(J_{[\alpha]}) = 3$ and that A is not LCM-stable over R. Note that $R = B + s\alpha B$ and R is a free B-module of rank 2. By making use of this fact, it is easy to check that $I_{[\alpha]} = (s, s\alpha + t)R$. Hence we have

$$J_{[\alpha]} = I_{[\alpha]} + \alpha I_{[\alpha]} = (s, t, u, s\alpha)R$$

which is a maximal ideal of height 3. Since s, t, u is a B-regular sequence and R is a free B-module, s, t, u is an R-regular sequence. On the other hand, we have $\alpha \notin (s, t, u)R$ and $(s\alpha)^2 \in (s, t, u)R$. Thus $\mathrm{grade}(J_{[\alpha]}) = 3$, as claimed. Recall that, in the case of a birational extension of integral domains, LCM-stability is equivalent to flatness (*cf.* [Ri]). Since A is not flat over R, we know that A is not LCM-stable over R.

Remark 4.3.15 H. Uda [U] proved:
If R is locally a GCD-domain, not necessarily Noetherian, then $R[\alpha]$ being LCM-stable over R is equivalent to $\mathrm{grade}(J_{[\alpha]}) \geq 3$. ∎

Using this, H. Uda presented the following example [U]:

Let $R := k[s, t, u]$ be a polynomial ring in three variables over a field k. Then $R[X]/(sX^2 + tX + u)$ is LCM-stable over R, but not flat over R.

Example 4.3.14 shows that Uda's theorem above does not hold in general without the assumption that R is locally a GCD-domain, even if R is Noetherian.

4.4 Some Subsets of Spec(R) in the High Degree Case

Let S be an R-algebra of finite type. A prime ideal $P \in \mathrm{Spec}(S)$ is called *unramified over* R (or S/R *unramified at* P) if $pS_P = PS_P$, where $p := P \cap R$, and $k(P)/k(p)$ is a finite separable extension. A prime ideal $p \in \mathrm{Spec}(R)$ is called *unramified in* S if all $P \in \mathrm{Spec}(S)$ lying over p are unramified over R (or S/R is unramified at p) (*cf.* [AK]). We say that S is *unramified* over R if every $P \in \mathrm{Spec}(S)$ is unramified over R. We recall that S is unramified over R if and only if the differential module $\Omega_R(S) = (0)$.

Let R be a Noetherian domain with the quotient field K and L be a finite algebraic extension field of K such that $[L : K] = d$. Let $\alpha \in L$. Let L be the quotient field of $R[\alpha]$.

When $R[\alpha]$ is an unramified extension of R, we call α an *unramified element* over R.

Let $\alpha \in L$ be an element unramified over R. For an ideal N of R, put $V(N) := \{p \in \mathrm{Spec}(R) | N \subseteq p\}$. Let $\Delta_{R[\alpha]/R} := \{p \in \mathrm{Spec}(R) | pR[\alpha] = R[\alpha]\}$ and let $\Gamma_{J_{[\alpha]}} := \{p \in \mathrm{Spec}(R) | p + J_{[\alpha]} = R\}$.

Recall that $\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})R$.

Proposition 4.4.1 *Assume that α is anti-integral over R. $\tilde{J}_{[\alpha]} = R$ if and only if the following two conditions hold:*

(1) *$R[\alpha]$ is a flat R-module;*

(2) *$\varphi : \mathrm{Spec}(R[\alpha]) \longrightarrow \mathrm{Spec}(R)$ is surjective.*

Proof ($\Leftarrow$) Suppose that $\tilde{J}_{[\alpha]} \neq R$. Then there exists a prime ideal p of R such that $p \supseteq \tilde{J}_{[\alpha]}$. Since φ is surjective, there exists a prime ideal P of $R[\alpha]$ such that $P \cap R = p$. Thus $\tilde{J}_{[\alpha]}R[\alpha] \subseteq P$.

We claim that $\tilde{J}_{[\alpha]}R[\alpha] = R[\alpha]$. Let a be any element of $I_{[\alpha]}$. Since $\varphi_\alpha(\alpha) = 0$, we have that $a\alpha^d + (a\eta_1)\alpha^{d-1} + \cdots + (a\eta_{d-1})\alpha = -a\eta_d \in \tilde{J}_{[\alpha]}R[\alpha]$. Hence $\tilde{J}_{[\alpha]}R[\alpha] \supseteq \tilde{J}_{[\alpha]}R[\alpha] + I_{[\alpha]}\eta_d \supseteq \tilde{J}_{[\alpha]} + I_{[\alpha]}\eta_d = J_{[\alpha]}$. Since $R[\alpha]$ is R-flat, we have that $J_{[\alpha]} = R$ by Theorem 2.2.5 or Proposition 2.5.4, and so $\tilde{J}_{[\alpha]}R[\alpha] \supseteq J_{[\alpha]}R[\alpha] = R[\alpha]$. Therefore $\tilde{J}_{[\alpha]}R[\alpha] = R[\alpha]$. Hence $P = R[\alpha]$, which is a contradiction. Thus $\tilde{J}_{[\alpha]} = R$.

($\Rightarrow$) If $\tilde{J}_{[\alpha]} = R$, then $J_{[\alpha]} = R$. By Theorem 2.2.5 or Proposition 2.5.5, we have that $R[\alpha]$ is R-flat. Now, by the following Theorem 4.4.2(1), $\mathrm{Im}(\varphi) =$

$D(\tilde{J}_{[\alpha]}) \cup V(J_{[\alpha]})$, we will have that $\mathrm{Im}(\varphi) = \mathrm{Spec}(R)$. Hence φ is surjective. $\square$

Theorem 4.4.2 *Assume that α is an anti-integral element of degree d over R. Then the following results hold:*

(1) $\mathrm{Im}(\varphi) = D(\tilde{J}_{[\alpha]}) \cup V(I_{[\alpha]}\eta_d) = D(\tilde{J}_{[\alpha]}) \cup V(J_{[\alpha]})$;
(2) $\Delta_{R[\alpha]/R} = V(\tilde{J}_{[\alpha]}) \cap \Gamma_{I_{[\alpha]}\eta_d} = V(\tilde{J}_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}}$.

Proof First, we shall prove (1). Let $p \in D(\tilde{J}_{[\alpha]})$ and $\pi : R[X] \to R[\alpha]$ be the canonical map. Then $R[\alpha]_p = R_p[X]/(\mathrm{Ker}(\pi))R_p[X]$ and

$$R[\alpha]_p/pR[\alpha]_p \cong (R_p/pR_p)[X]/(\overline{\mathrm{Ker}}(\pi))(R_p/pR_p)[X]$$

where $\overline{\mathrm{Ker}(\pi)}$ denotes the image of $\mathrm{Ker}(\pi)$ in $(R_p/pR_p)[X]$. Since $p \not\supseteq \tilde{J}_{[\alpha]}$, $\overline{\mathrm{Ker}(\pi)}$ is not 0. Thus $pR[\alpha]_p \neq R[\alpha]_p$. Let $\widetilde{P}$ be a prime divisor of $pR[\alpha]_p$. Put $\widetilde{P} \cap R[\alpha] = P$. Then $\widetilde{P} \cap R_p = pR_p$ and so $P \cap R = p$. Hence $p \in \mathrm{Im}(\varphi)$. Next, let $p \supseteq \eta_d I_{[\alpha]}$. We recall that if $p \not\supseteq \tilde{J}_{[\alpha]}$ then $p \in \mathrm{Im}(\varphi)$. So, we suppose that $p \supseteq \tilde{J}_{[\alpha]}$. Then $p \supseteq J_{[\alpha]}$. It follows that $R[\alpha]/pR[\alpha] \cong (R/p)[T]$, where T is an indeterminate. Since $pR[X] \supseteq \mathrm{Ker}(\pi)$, $P = pR[\alpha] \in \mathrm{Spec}(R[\alpha])$, and $P \cap R = p$, we have that $p \in \mathrm{Im}(\varphi)$. Thus we proved that $D(\tilde{J}_{[\alpha]}) \cup V(J_{[\alpha]}) \subseteq D(\tilde{J}_{[\alpha]}) \cup V(I_{[\alpha]}\eta_d) \subseteq \mathrm{Im}(\varphi)$. To prove the opposite inclusion, assume that $p \notin D(\tilde{J}_{[\alpha]}) \cup V(I_{[\alpha]}\eta_d)$. We claim that $p \notin \mathrm{Im}(\varphi)$. Suppose that $p \in \mathrm{Im}(\varphi)$. There exists a prime ideal P of $R[\alpha]$ such that $P \cap R = p$. Since $p \not\supseteq \eta_d I_{[\alpha]}$, there exists some element $a \in I_{[\alpha]}$ such that $a\eta_d \notin p$. Since $p \supseteq \tilde{J}_{[\alpha]} \ni a, a\eta_1, \ldots, a\eta_{d-1}$, it follows that $P \ni (a\alpha^d + a\eta_1\alpha^{d-1} + \cdots + a\eta_{d-1}\alpha) = -a\eta_d$. So $p \ni -a\eta_d$. This is a contradiction. We have that $p \notin \mathrm{Im}(\varphi)$. Hence we have that $\mathrm{Im}\ \varphi = D(\tilde{J}_{[\alpha]}) \cup V(I_{[\alpha]}\eta_d)$. We claim that $D(\tilde{J}_{[\alpha]}) \cup V(I_{[\alpha]}\eta_d) = D(\tilde{J}_{[\alpha]}) \cup V(J_{[\alpha]})$. Clearly, it follows that $D(\tilde{J}_{[\alpha]}) \cup V(I_{[\alpha]}\eta_d) \supseteq D(\tilde{J}_{[\alpha]}) \cup V(J_{[\alpha]})$. To prove the opposite inclusion, assume that $p \notin D(\tilde{J}_{[\alpha]}) \cup V(J_{[\alpha]})$. Then $p \supseteq \tilde{J}_{[\alpha]}$ and $p \not\supseteq J_{[\alpha]}$. Suppose that $p \in V(I_{[\alpha]}\eta_d) \cup D(\tilde{J}_{[\alpha]})$. Then $p \supseteq I_{[\alpha]}\eta_d + \tilde{J}_{[\alpha]} = J_{[\alpha]}$.This is a contradiction. Therefore we have that $p \notin V(I_{[\alpha]}\eta_d) \cup D(\tilde{J}_{[\alpha]})$. Thus we complete the proof of the claim.
(2) Let $p \in V(\tilde{J}_{[\alpha]}) \cap \Gamma_{I_{[\alpha]}\eta_d}$. Suppose that $p \notin \Delta_{R[\alpha]/R}$. Then there exists a prime ideal P of $R[\alpha]$ such that $P \supseteq pR[\alpha]$. Put $P \cap R = q$. Then we have that $q \supseteq p$ and $q \in \mathrm{Im}(\varphi)$. Since $a,\ a\eta_1,\ \ldots, a\eta_{d-1} \in \tilde{J}_{[\alpha]} \subseteq p \subseteq P$, for any $a \in I_{[\alpha]}$, we have that

$$a\alpha^d + a\eta_1\alpha^{d-1} + \cdots + a\eta_{d-1}\alpha = -a\eta_d \in P \cap R = q$$

Consequently, we have that $I_{[\alpha]}\eta_d \subseteq q$. Since $p \in \Gamma_{I_{[\alpha]}\eta_d}$, it follows that $q \supseteq I_{[\alpha]}\eta_d + p = R$. This is a contradiction. Therefore we proved that $V(\tilde{J}_{[\alpha]}) \cap$

$\Gamma_{I_{[\alpha]}\eta_d} \subseteq \Delta_{R[\alpha]/R}$. Conversely, we shall prove that $\Delta_{R[\alpha]/R} \subseteq V(\tilde{J}_{[\alpha]}) \cap \Gamma_{I_{[\alpha]}\eta_d}$. Let $p \notin V(\tilde{J}_{[\alpha]}) \cap \Gamma_{I_{[\alpha]}\eta_d}$. Then $p \not\supseteq \tilde{J}_{[\alpha]}$ or $p + I_{[\alpha]}\eta_d \neq R$. If $p \not\supseteq \tilde{J}_{[\alpha]}$, then $p \in \mathrm{Im}(\varphi)$ by Theorem 4.4.2(1). Therefore there exists a prime ideal P of $R[\alpha]$ such that $P \cap R = p$. Suppose that $p \in \Delta_{R[\alpha]/R}$. Then $P \supseteq pR[\alpha] = R[\alpha]$. This is a contradiction. Thus $p \notin \Delta_{R[\alpha]/R}$. Next, if $p + I_{[\alpha]}\eta_d \neq R$, then there exists a prime ideal q of R such that $p + I_{[\alpha]}\eta_d \subseteq q$. Since $q \in \mathrm{Im}(\varphi)$ by Theorem 4.4.2(1), it follows that $qR[\alpha] \neq R[\alpha]$. Since $p \subseteq q$, we have that $pR[\alpha] \neq R[\alpha]$. So we get $p \notin \Delta_{R[\alpha]/R}$. Hence we complete the proof of $\Delta_{R[\alpha]/R} = V(\tilde{J}_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}}$. At last, we shall prove that $V(\tilde{J}_{[\alpha]}) \cap \Gamma_{I_{[\alpha]}\eta_d} = V(\tilde{J}_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}}$. Clearly, we have that $V(\tilde{J}_{[\alpha]}) \cap \Gamma_{I_{[\alpha]}\eta_d} \subseteq V(\tilde{J}_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}}$. So we shall prove that $V(\tilde{J}_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}} \subseteq V(\tilde{J}_{[\alpha]}) \cap \Gamma_{I_{[\alpha]}\eta_d}$. Let $p \in V(\tilde{J}_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}}$. Suppose that $p \notin V(\tilde{J}_{[\alpha]}) \cap \Gamma_{I_{[\alpha]}\eta_d}$. Then $p + \tilde{I}_{[\alpha]} \neq R$. Also there exists a prime ideal q of R such that $q \supseteq p + I_{[\alpha]}\eta_d$. Since $q \supseteq J_{[\alpha]}$ and $p + J_{[\alpha]} = R$, we have that $q \supseteq p + J_{[\alpha]} = R$. This is a contradiction. Hence $p \in V(\tilde{J}_{[\alpha]}) \cap \Gamma_{I_{[\alpha]}\eta_d}$. This completes the proof. $\square$

Chapter 5

Unramified Extensions

Throughout this chapter, the following notations will be in force unless otherwise specified:

R: a Noetherian integral domain,

$K := K(R)$: the quotient field of R,

L: an algebraic field extension of K,

α: a nonzero element of L,

$d = [K(\alpha) : K]$,

$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$, the minimal polynomial of α over K.

$I_{[\alpha]} := \bigcap_{i=1}^d (R :_R \eta_i) = \bigcap_{i=1}^d I_{\eta_i}$, which is an ideal of R.

$I_a := R :_R aR$ for $a \in K$.

It is clear that for $a \in K$, $I_{[a]}$ stands for I_a from definitions.

$$J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d),$$
$$\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}).$$

We also use the following standard notation:

$$\mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) | \mathrm{depth}(R_p) = 1\}.$$

Let A be an R-algebra. Then

$\Omega_R(A)$ denotes the universal module of differentials of A over R.

5.1 Unramifiedness and Etaleness of Super-Primitive Extensions

Let S be an R-algebra of finite type. A prime ideal $P \in \mathrm{Spec}(S)$ is called *unramified over* R (or S/R *unramified at* P) if $pS_P = PS_P$, where $p := P \cap R$, and $k(P)/k(p)$ is a finite separable algebraic extension. A prime ideal

$p \in \mathrm{Spec}(R)$ is called *unramified in* S if all $P \in \mathrm{Spec}(S)$ lying over p are unramified over R (or S/R is unramified at p) (*cf.* [AK]).

It is known that $\Omega_R(S) = 0$ if and only if S is unramified over R. The following result can be proved by using [AK, V1(6.8)] but we give a direct proof. If α is super-primitive and integral over R, $R[\alpha]$ is finite, flat over R (*cf.* Theorems 2.3.2 and 2.3.6.).

Proposition 5.1.1 *Assume that α is an anti-integral element and is integral over R. Then $R[\alpha]$ is unramified over R if and only if $R[\alpha]_p$ is unramified over R_p for each $p \in \mathrm{Dp}_1(R)$.*

Proof Since $R[\alpha]$ is integral over R, $\varphi_\alpha(X) \in R[X]$ by Theorem 2.3.2. For a polynomial f, we denote the derivative of f by f'. Then $\varphi'_\alpha(\alpha) = d\alpha^{d-1} + (d-1)\eta_1\alpha^{d-2} + \cdots + \eta_{d-1}$ and let $p \in \mathrm{Spec}(R)$. Then $\varphi'_\alpha(\alpha)R[\alpha] \not\subseteq P$ for any $P \in \mathrm{Spec}(R[\alpha])$ with $P \cap R = p$ if and only if $R[\alpha]_p$ is unramified over R_p (*cf.* [AK, V1(6.12)]). Suppose that $\varphi'_\alpha(\alpha)R[\alpha] \neq R[\alpha]$. Then there exists $P \in \mathrm{Ht}_1(R[\alpha])$ such that $\varphi'_\alpha(\alpha) \in P$. Put $p = P \cap R$. Then $\mathrm{depth}(R[\alpha]_P) = 1$ implies $\mathrm{depth}(R_p) = 1$ because $R[\alpha]_p$ is flat over R_p. Thus $R[\alpha]_p$ is unramified over R_p by the assumption. Hence $R[\alpha]_P$ is unramified over R_p, which is a contradiction. So $\varphi'_\alpha(\alpha)R[\alpha] = R[\alpha]$, which means that $R[\alpha]$ is unramified over R. $\square$

Remark 5.1.2 Let the notation be the same as in Proposition 5.1.1 and its proof. Let $B = R[\alpha, \alpha^{-1}]$. Then for $P \in \mathrm{Spec}(B)$, B_P is unramified over $R_{R\cap P}$ if and only if $P \not\supseteq \varphi'_\alpha(\alpha)B$. Indeed, let $P \subseteq B$ be a prime ideal and put $Q = P \cap R[\alpha]$ and $p = P \cap R$. When B_P/R_p is ramified, $R[\alpha]_Q/R_p$ is ramified. So $\varphi'_\alpha(\alpha) \in Q \subseteq P$. Conversely, if $\varphi'_\alpha(\alpha) \in P$, then $Q = P \cap R[\alpha] \ni \varphi'_\alpha(\alpha)$. So $B_P = R[\alpha]_Q$ is ramified over R_p. $\square$

It is known that the purity of branch locus holds for a finite flat extension [AK]. The following is a result similar to this fact.

Proposition 5.1.3 *Assume that α is a super-primitive element and is flat over R and that R contains an infinite field k. Then $R[\alpha]$ is unramified over R if and only if $R[\alpha]_p$ is unramified over R_p for any $p \in \mathrm{Dp}_1(R)$.*

Proof We have only to consider the case that R is a local ring. So we may assume that (R, m) is a local ring. If $R[\alpha]$ is integral over R, we have shown this in Proposition 5.1.1. Assume that $R[\alpha]$ is not integral over R. Since $J_{[\alpha]} = R$ by Theorem 2.3.6, replacing α by $\alpha - \lambda$ for some $\lambda \in k$, we may assume by

Proposition 2.2.10 that α satisfies that

(a) $\alpha^{-1} \in R[\alpha]$;

(b) α^{-1} is a super-primitive element of degree d over R;

(c) α^{-1} is integral over R.

Hence we have

$$R \subseteq R[\alpha^{-1}] \subseteq R[\alpha, \alpha^{-1}] = R[\alpha]$$

Apply Remark 5.1.2 to $B = R[\alpha^{-1}][(\alpha^{-1})^{-1}] = R[\alpha]$. We conclude that for $P \in \mathrm{Spec}(R[\alpha])$, $R[\alpha]_P$ is unramified over $R_{P\cap R}$ if and only if $P \not\supseteq \varphi'_{\alpha^{-1}}(\alpha^{-1})R[\alpha]$. In the same way as in the proof of Proposition 5.1.1, the assumption that $R[\alpha]_p$ is unramified over R_p for any $p \in \mathrm{Dp}_1(R)$ yields that $R[\alpha]$ is unramified over R. $\square$

Let S denote an R-algebra of finite type. We say that S is *etale* over R if S is flat and unramified over R (*cf.* [AK]).

As a consequence of Propositions 5.1.1 and 5.1.3, we obtain the following theorem.

Theorem 5.1.4 *Assume that α is a super-primitive element over R and that R contains an infinite field k. Then there exist $p_1, \ldots, p_t \in \mathrm{Dp}_1(R)$ (t may be 0) such that the nonetale locus of $R[\alpha]$ over R is given by $V(J_{[\alpha]}) \cup (\bigcup_{i=1}^{t} V(p_i))$.*

Example 5.1.5 Let k be a field, a, b indeterminates, and R = $k[a, b]$. Let α be a root of an equation $aX^2 + bX + a = 0$. Then $J_{[\alpha]} = (a, b)R$. Assume that $p \in \mathrm{Spec}(R)$ and $p \not\supseteq J_{[\alpha]}$. When $a \notin p$, $(2\alpha + b/a)R[\alpha]_p$ is the ramification locus. When $a \in p$ and $b \notin p$, $(\alpha + 1)R[\alpha]_p$ is the ramification locus.

Definition 5.1.6 Let A be an extension of R with $[K(A) : K] = d$. Define

$$\Lambda(A) := \{q \in \mathrm{Spec}(R) | \mathrm{rank}_{k(q)} A \otimes_R k(q) = d\}$$

It is easy to see that when α is a super-primitive element of degree d over R, we have:

$\Lambda(R[\alpha]) \supseteq \mathrm{Dp}_1(R) \Leftrightarrow R[\alpha]$ is integral over $R \Rightarrow R[\alpha]$ is flat over R.

When A is a finitely generated extension of R, define:

$$Ur(A) := \{p \in \mathrm{Spec}(R) | A_p \text{ is unramified over } R_p\}$$

which is an open set of $\mathrm{Spec}(R)$.

Under these preparations, we finally obtain the following.

Theorem 5.1.7 *Assume that $[L : k] = d$, and that $\alpha_1, \ldots, \alpha_n \in L$ are super-primitive elements of degree d, and let $A = R[\alpha_1, \ldots, \alpha_n]$. If $\Lambda(R[\alpha_i]) \supseteq \mathrm{Dp}_1(R)$ $(1 \leq i \leq n)$ and $Ur(R[\alpha_j]) \supseteq \mathrm{Dp}_1(R)$ for some j, then A is integral over R, and A_p is etale over R_p for any $p \in \Lambda(A)$. If $\Lambda(A) = Spec(R)$ in addition to the preceding assumptions, then A is integral and etale over R.*

Proof The assumption $\mathrm{Dp}_1(R) \subseteq \Lambda(R[\alpha_i])$ implies that α_i is integral over R and $\Lambda(R[\alpha_i]) = \mathrm{Spec}(R)$ by Theorem 2.3.2, and hence A is integral over R. Take $p \in \Lambda(A)$. Then $p \in \Lambda(R[\alpha_j])$ and $R[\alpha_j]$ is finite, flat over R as was shown in Theorem 2.2.5. Thus $R[\alpha_j]_p$ is an R_p-free module of rank d. Since $Ur(R[\alpha_j]) \supseteq \mathrm{Dp}_1(R)$, $R[\alpha_j]$ is unramified over R by Proposition 5.1.1. Hence $pR[\alpha_j]_p$ is a radical ideal. Noting that A is integral over $R[\alpha_j]$, we have $pA_p \cap R[\alpha_j]_p = pR[\alpha_j]_p$. Thus $R[\alpha_j]_p/pR[\alpha_j]_p \subseteq A_p/pA_p$. As both sides have the same dimension d as vector spaces over $k(p)$, we have $R[\alpha_j]_p/pR[\alpha_j]_p = A_p/pA_p$, which means that $A_p = R[\alpha_j]_p + pA_p$. By Nakayama's lemma, we get $A_p = R[\alpha_j]_p$. Therefore A_p is unramified and flat (i.e., etale) over R_p for any $p \in \Lambda(A)$. ∎

5.2 Differential Modules of Anti-Integral Extensions

In this section, we treat the module of differentials. The definition of such modules are seen in [M2] and [AK].

Theorem 5.2.1 *Assume that α is an anti-integral element of degree d over R. Let $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ be the monic minimal polynomial of α over K. Then the following conditions are equivalent:*

(1) $\Omega_R(R[\alpha]) = (0)$;

(2) $I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha] = R[\alpha]$.

Proof Put $N = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Let $0 \longrightarrow N \longrightarrow R[X] \longrightarrow R[X]/N \longrightarrow 0$ be an exact sequence. From this exact sequence, we get the following exact sequence:

$$N/N^2 \longrightarrow R[\alpha] \otimes_{R[X]} R[X]dX \cong R[\alpha] \longrightarrow \Omega_R(R[\alpha]) \longrightarrow 0$$

where $\rho : N/N^2 \longrightarrow R[\alpha] \otimes_{R[X]} R[X]dX$ is a map such that $\rho(x \bmod N^2) = d_{R[X]/R}x \otimes 1$ for $x \in N$. Note that $R[\alpha] \bigotimes_{R[X]} \Omega_R(R[X]) \cong R[\alpha]$. Since $N =$

$I_{[\alpha]}\varphi_\alpha(X)R[X]$, it follows that $\mathrm{Im}(\rho) = I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha]$. Hence $\Omega_R(R[\alpha]) = (0)$ if and only if $I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha] = R[\alpha]$. ∎

Example 5.2.2 Let k be a field with characteristic $\neq 2$ and let $R = k[X^2, 1/X^2]$ be a subring of $k[X, 1/X]$. Put $\alpha := X$. Then $K = k(X^2)$ is the quotient field of R and $L = k(X)$ is the quotient field of $R[\alpha]$. Then $\varphi_\alpha(Y) = Y^2 - X^2 \in R[Y]$ is the monic minimal polynomial of α over K. Since $\varphi'_\alpha(Y) = 2Y$, $\varphi'_\alpha(X) = 2X$ is a unit of $R[\alpha] = k[X, 1/X]$. Hence $R[\alpha]$ is unramified over R (*cf.* [N]).

In the rest of this section, we use the notation of Theorem 5.2.1 unless otherwise specified.

Remark 5.2.3 Let α denote an anti-integral element of degree d over R. If $R[\alpha]$ is unramified over R, then $\tilde{J}_{[\alpha]}R[\alpha] = I_{[\alpha]}R[\alpha] = R[\alpha]$. In fact, since $I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha] \subseteq I_{[\alpha]}R[\alpha]$ and $I_{[\alpha]} \subseteq \tilde{J}_{[\alpha]}$, it follows that $I_{[\alpha]}R[\alpha] = \tilde{J}_{[\alpha]}R[\alpha] = R[\alpha]$. ∎

Theorem 5.2.4 *Assume that α is an anti-integral element over R. If $\Omega_R(R[\alpha]) = (0)$, then $R[\alpha]$ is a flat R-module.*

Proof Suppose that $R[\alpha]$ is not a flat R-module. Then there exists a prime ideal p of R such that $p \supseteq \mathrm{c}(I_{[\alpha]}\varphi_\alpha(X))$ by Theorem 2.2.5. Since $pR[X] \supseteq I_{[\alpha]}\varphi_\alpha(X)R[X] = N$, we have that $R[\alpha]/pR[\alpha] \cong (R/p)[X]$. Hence $pR[\alpha] \in \mathrm{Spec}(R[\alpha])$. From an exact sequence $0 \longrightarrow pR[\alpha] \longrightarrow R[\alpha] \longrightarrow R[\alpha]/pR[\alpha] \longrightarrow 0$, we have the following exact sequence: $pR[\alpha]/p^2R[\alpha] \longrightarrow \Omega_R(R[\alpha]) \bigotimes_{R[\alpha]} R[\alpha]/pR[\alpha] \longrightarrow \Omega_{R/p}(R[\alpha]/pR[\alpha]) \longrightarrow 0$. But $\Omega_R(R[\alpha]) = (0)$, and so $\Omega_{R/p}(R[\alpha]/pR[\alpha]) = (0)$. Also, since $R[\alpha]/pR[\alpha] \cong (R/p)[X]$, we have that $\Omega_{R/p}(R[\alpha]/pR[\alpha]) = ((R/p)[X])dX \neq (0)$. This is a contradiction. Hence $R[\alpha]$ is a flat R-module. ∎

Remark 5.2.5 Let α be an anti-integral over R. Although $R[\alpha]$ is a flat R-module, $R[\alpha]$ is not necessarily unramified over R. For example, let $R = k[X^2]$ be a subring of $k[X]$ where k denotes a field with characteristic $\neq 2$. Set $R[\alpha] = R[X] = k[X]$. Since $\Omega_R(R[\alpha]) = R[\alpha]/XR[\alpha] \neq (0)$, we have that $R[\alpha]$ is not an unramified extension over R. ∎

Proposition 5.2.6 *Under the assumption in Theorem 5.2.1, it follows that $V(\mathrm{Ann}_R(\Omega_R(R[\alpha]))) \supseteq V(J_{[\alpha]})$, where $\mathrm{Ann}_R(\Omega_R(R[\alpha]))$ denotes the annihilator ideal of $\Omega_R(R[\alpha])$.*

Proof Assume that $p \notin V(\mathrm{Ann}_R(\Omega_R(R[\alpha])))$. So there exists an element $s \in \mathrm{Ann}_R(\Omega_R(R[\alpha]))$ such that $s \notin p$. Therefore $s\Omega_R(R[\alpha]) = (0)$ and so

$\Omega_{R_p}(R[\alpha]_p) = (0)$. Note that $R[\alpha]_p = R_p[\alpha]$. Using Theorem 5.2.4, it follows that $R[\alpha]_p$ is a flat R_p-module. From this fact and Theorem 2.2.5, we obtain that $p \not\supseteq J_{[\alpha]}$. Therefore $p \notin V(J_{[\alpha]})$. ∎

Proposition 5.2.7 *The element α is an anti-integral element of degree d over R if and only if α is an anti-integral element of degree d over R_p for any $p \in \mathrm{Spec}(R)$.*

Proof Let $0 \longrightarrow N \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0$ be an exact sequence.
($\Rightarrow$) N is an ideal generated by some polynomials of degree d, where $d = [L : K]$. Also,

$$0 \longrightarrow N_p \longrightarrow R_p[X] \longrightarrow R_p[\alpha] \longrightarrow 0$$

is an exact sequence. Hence N_p is also an ideal generated by some polynomials of degree d. Thus α is an anti-integral element of degree d over R_p.
($\Leftarrow$) Put $B = J_{[\alpha]}\varphi_\alpha(X)R[X]$. Then $B \subseteq N$. And $N = B$ if and only if α is anti-integral over R. By the assumption, we have that $N_p = B_p$ for any $p \in \mathrm{Spec}(R)$. Hence we get $N = B$. ∎

Theorem 5.2.8 *If α is an anti-integral element in K, then the following four statements are equivalent:*

(i) $I_\alpha R[\alpha] = R[\alpha]$;
(ii) $R[\alpha]$ *is a flat R-module;*
(iii) $J_{[\alpha]}R[\alpha] = R[\alpha]$;
(iv) $R[\alpha]$ *is an unramified extension of R.*

Proof Using the fact that $I_\alpha R[\alpha] = J_{[\alpha]}R[\alpha]$ and Theorem 1.2.8, the equivalence of (i), (ii), and (iii) are already proved. Also, it can be proved that (iv) implies (ii) from Theorem 5.2.4. Now, we claim that (ii) implies (iv). Since $d = 1$, we have that $J_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha] = J_{[\alpha]}R[\alpha]$, where $\varphi_\alpha(X) = X - \alpha$. Since $R[\alpha]$ is a flat R-module, it follows that $J_{[\alpha]} = R$. So we get that $J_{[\alpha]}R[\alpha] = R[\alpha]$. Therefore we have that $I_\alpha\varphi'_\alpha(\alpha)R[\alpha] = J_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha] = R[\alpha]$. Using Theorem 5.2.1, we see that $\Omega_R(R[\alpha]) = (0)$. Hence $R[\alpha]$ is an unramified extension of R. ∎

Remark 5.2.9 In Theorem 5.2.8, when $\alpha \in K$ is anti-integral over R, that is, a simple birational anti-integral extension is flat if and only if it is an unramified extension, but, in case of a nonbirational extension, flatness is not equivalent to unramifiedness (*cf.* Remark 5.2.5). ∎

Remark 5.2.10 Assume that α is anti-integral over R. Let $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ be the monic minimal polynomial of α over K, where $d = [L : K] > 1$. Then $I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha] \subseteq \tilde{J}_{[\alpha]}R[\alpha]$, for $I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha] = I_{[\alpha]}(d\alpha^{d-1} + (d-1)\eta_1\alpha^{d-2} + \cdots + \eta_{d-1})R[\alpha] \subseteq I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})R[\alpha] \subseteq \tilde{J}_{[\alpha]}R[\alpha]$. From Theorem 5.2.1, if $\Omega_R(R[\alpha]) = (0)$, then $\tilde{J}_{[\alpha]}R[\alpha] = R[\alpha]$. As an example of $I_{[\alpha]}\varphi'_\alpha(\alpha) \neq \tilde{J}_{[\alpha]}R[\alpha]$, we give the following example. Put $\varphi_\alpha(X) = X^2 - a \in R[X]$ where a is not a unit of R. Then $I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha] = 2\alpha R[\alpha] \neq R[\alpha]$ and $\tilde{J}_{[\alpha]}R[\alpha] = R[\alpha]$ (we assume that 2 is not a unit of $R[\alpha]$). ∎

Proposition 5.2.11 *Assume that $R[\alpha]$ is an unramified extension of R and let $\varphi : \mathrm{Spec}(R[\alpha]) \longrightarrow \mathrm{Spec}(R)$ be a restriction map. Then φ is surjective if and only if $\tilde{J}_{[\alpha]} = R$.*

Proof ($\Leftarrow$): When $\tilde{J}_{[\alpha]} = R$, $R = \tilde{J}_{[\alpha]} \subseteq J_{[\alpha]}$ and hence α is a super-primitive element over R. From Theorem 4.3.2, we know that $\mathrm{Im}\,(\varphi) = D(\tilde{J}_{[\alpha]}) \cup V(J_{[\alpha]})$. So $\mathrm{Im}(\varphi) = \mathrm{Spec}(R)$ by assumption. Hence φ is surjective.
($\Rightarrow$): Suppose that $\tilde{J}_{[\alpha]} \neq R$. There exists a prime ideal p of R such that $\tilde{J}_{[\alpha]} \subseteq p$. Since φ is surjective, there exists a prime ideal P of $R[\alpha]$ such that $P \cap R = p$. Thus $\tilde{J}_{[\alpha]}R[\alpha] \subseteq P$. But $R[\alpha]$ is an unramified extension over R, $\tilde{J}_{[\alpha]}R[\alpha] = R[\alpha]$ from Theorem 5.2.1 and Remark 5.2.3. Hence $R[\alpha] = P$, a contradiction. Therefore $\tilde{J}_{[\alpha]} = R$. ∎

Remark 5.2.12 Assume that α is anti-integral over R. When φ is surjective, for an ideal N of R, $NR[\alpha] = R[\alpha]$ if and only if $N = R$. ∎

Proposition 5.2.13 *Let A be a ring extension of R and N be an ideal of R. Let $N = q_1 \cap q_2 \cap \ldots \cap q_n$ be a primary decomposition of N, where $\sqrt{q_i} = p_i$ for $1 \leq i \leq n$. Then $NA = A$ if and only if $p_i A = A$ for $1 \leq i \leq n$.*

Proof ($\Rightarrow$) Since $NA \subseteq p_i A$, we have $p_i A = A$.
($\Leftarrow$) From $p_i A = A$, we have that $1 = \Sigma_j\, a_{ij}\alpha_{ij}$ ($a_{ij} \in p_i, \alpha_{ij} \in A$). Clearly, it can be assumed that $a_{ij} \in q_i$. Then $1 = \prod_i (\sum a_{ij}\alpha_{ij}) \in NA$ and so $NA = A$. ∎

Theorem 5.2.14 *Assume that α is an anti-integral element of degree d over R. $R[\alpha]$ is a flat R-module if and only if $\tilde{J}_{[\alpha]}R[\alpha] = R[\alpha]$.*

Proof Let $\tilde{J}_{[\alpha]} = q_1 \cap q_2 \cap \cdots \cap q_n$ ($\sqrt{q_i} = p_i$) be a primary decomposition of $\tilde{J}_{[\alpha]}$. ($\Rightarrow$) From Theorem 4.3.2 (2), $\Delta_{R[\alpha]/R} = V(\tilde{J}_{[\alpha]}) \cap \Gamma_{\eta_d I_{[\alpha]}}$, where $J_{[\alpha]} =$

$\tilde{J}_{[\alpha]} + \eta_d I_{[\alpha]}$. Since $R[\alpha]$ is R-flat, it follows that $J_{[\alpha]} = R$ from Theorem 2.3.6. Since $\tilde{J}_{[\alpha]} \subseteq p_i$, we get that $p_i \in V(\tilde{J}_{[\alpha]})$ and $p_i \in \Gamma_{\eta_d I_{[\alpha]}}$. Therefore $p_i \in V(\tilde{J}_{[\alpha]}) \cap \Gamma_{\eta_d I_{[\alpha]}} = \Delta_{R[\alpha]/R}$ and so $p_i R[\alpha] = R[\alpha]$. From Proposition 5.2.13, we have $\tilde{J}_{[\alpha]} R[\alpha] = R[\alpha]$. ($\Leftarrow$) Since $\tilde{J}_{[\alpha]} R[\alpha] = R[\alpha]$, it follows that $p_i R[\alpha] = R[\alpha]$ from Proposition 5.2.13. So $p_i \in \Delta_{R[\alpha]/R} = V(\tilde{J}_{[\alpha]}) \cap \Gamma_{\eta_d I_{[\alpha]}}$. Since $p_i + \eta_d I_{[\alpha]} = R$, we have that $J_{[\alpha]} = \eta_d I_{[\alpha]} + \tilde{J}_{[\alpha]} = R$. Hence $R[\alpha]$ is a flat R-module. ∎

We recall that α is called an unramified element over R if $R[\alpha]$ is an unramified extension of R.

Theorem 5.2.15 *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in L$ be unramified elements over R. Then $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ is an unramified extension over R.*

Proof Clearly, it can be assumed that $n = 2$. Let $B = R[\alpha_1]$, $C = R[\alpha_2]$, and $A = B[\alpha_2] = R[\alpha_1, \alpha_2]$. Then we have the following exact sequence [M1]:

$$\Omega_R(B) \otimes_B A \longrightarrow \Omega_R(A) \longrightarrow \Omega_B(A) \longrightarrow 0$$

Since B and C are unramified extensions over R, we see that $\Omega_R(B) = (0)$ and $\Omega_R(C) = (0)$. So it suffices to show that $\Omega_B(A) = (0)$. Since C is unramified over R, it follows that $B \otimes_R C$ is unramified over B, that is, $\Omega_B(B \otimes_R C) = (0)$. From the exact sequence $0 \longrightarrow N \longrightarrow B \otimes_R C \longrightarrow A \longrightarrow 0$, and the fact that $\Omega_B(A) = \Omega_B(B \otimes_R C)/\Omega(N)$ where $\Omega(N)$ denotes the submodule generated by $\{da \mid a \in N\}$, we get that $\Omega_B(A) = (0)$. Thus we have that $\Omega_R(A) = (0)$. The proof is complete. ∎

Remark 5.2.16 Assume that α is an element of the quotient field of R and that $R[\alpha]$ is an anti-integral extension of R. Then $R[\alpha]$ is a flat R-module if and only if for each $p \in \mathrm{Spec}(R)$, $R[\alpha]_p = R_p$, or $R[\alpha]_p = R_p[1/a]$ for some element $a \in R$. ∎

Proof ($\Leftarrow$) It is trivial to show this implication.
($\Rightarrow$) In the case $p \not\supseteq I_\alpha$, it follows that $R[\alpha]_p = R_p$. In the remaining case, since $J_{[\alpha]} = I_\alpha + \alpha I_\alpha = R$, we have that $p \not\supseteq \alpha I_\alpha$. And so there exists an element a of I_α such that $a\alpha \notin p$. Put $b = a\alpha$. Then $R[\alpha]_p = R_p[\alpha] = R_p[1/a]$. ∎

In the rest of this section, we examine the ramification locus of $R\langle\alpha\rangle$ over R when α is an anti-integral element of degree $d > 1$ over R.

Proposition 5.2.17 *Assume that α is an anti-integral element of degree $d > 1$ over R. If $R\langle\alpha\rangle$ is unramified over R, then $I_{[\alpha]} = R$ and hence $R[\alpha]$ is integral over R.*

Proof Suppose that $I_{[\alpha]} \neq R$. Then there exists a prime ideal p of R such that $p \supseteq I_{[\alpha]}$. By Proposition 3.1.25 (1), $P := p \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ is a prime ideal of B. By construction, P is a unique prime ideal lying over p and $B/P = R/p$. Since B is unramified over R, pB is a radical ideal. Since B is integral over R, we have $pB = P$. Hence $B = R + pB$. Since B is a finite R-module by Theorem 3.1.18, localizing at p and using Nakayama's lemma, we have $B = R$. But $B \neq R$ because $d > 1$ by assumption. Thus $I_{[\alpha]} = R$. The last statement follows from Theorem 2.3.2. ∎

Remark 5.2.18 Assume that $I_{[\alpha]} = R$. Since α is anti-integral and integral over R, we have $R[\alpha] = R\langle\alpha\rangle$, which is a free R-module of rank d. Moreover $R[\alpha]$ is unramified over R if and only if $\varphi'_\alpha(\alpha)$ is an invertible element in $R[\alpha]$, where $\varphi'_\alpha(X)$ denotes the derivative of $\varphi_\alpha(X)$. ∎

Proposition 5.2.19 *Assume that α is an anti-integral element of degree $d > 1$ over R. Then for each prime ideal p of R, the following statements are equivalent:*

(i) *$R\langle\alpha\rangle_p$ is ramified over R_p;*

(ii) *either $I_{[\alpha]} \subseteq p$ holds or both $I_{[\alpha]} \nsubseteq p$ and $\varphi'_\alpha(\alpha)R\langle\alpha\rangle_p \neq R\langle\alpha\rangle_p$ hold.*

Proof (i) $\Rightarrow$ (ii): Suppose that $I_{[\alpha]} \nsubseteq p$ and $\varphi'_\alpha(\alpha)R\langle\alpha\rangle_p = R\langle\alpha\rangle_p$. Since $I_{[\alpha]} \nsubseteq p$, we have $R\langle\alpha\rangle_p = R_p[\alpha]$. Since $\varphi'_\alpha(\alpha)R\langle\alpha\rangle_p = R\langle\alpha\rangle_p$, we have $\varphi'_\alpha(\alpha)R_p[\alpha] = R_p$. Thus $R\langle\alpha\rangle_p$ is unramified over R_p.
(ii) $\Rightarrow$ (i): Note that $R\langle\alpha\rangle_p = R_p[\alpha] \cap R_p[\alpha^{-1}]$ and that α is anti-integral over R_p of degree d. By Proposition 5.2.17, the unramifiedness of $R\langle\alpha\rangle_p/R_p$ yields $I_{[\alpha]} \nsubseteq p$. By Remark 5.2.18, we have $\varphi'_\alpha(\alpha)R\langle\alpha\rangle_p = R\langle\alpha\rangle_p$. ∎

Finally we determine the ramification locus of $R\langle\alpha\rangle$ over R.

Theorem 5.2.20 *Assume that α is an anti-integral element of degree $d > 1$ over R. Then the ramification locus of $R\langle\alpha\rangle$ over R is given by $I_{[\alpha]}^d\varphi'_\alpha(\alpha)R\langle\alpha\rangle \cap R$. Consequently, $\sqrt{\mathrm{Ann}_R(\Omega_R(R\langle\alpha\rangle))} = \sqrt{I_{[\alpha]}^d\varphi'_\alpha(\alpha)R\langle\alpha\rangle \cap R}$.*

Proof By Theorem 3.1.6, we have $R\langle\alpha\rangle :_R \alpha = I_{[\alpha]}$. Since $\varphi'_\alpha(\alpha)$ is regarded as a polynomial of α of degree $d-1$, it follows that $I_{[\alpha]}^{d-1}\varphi'_\alpha(\alpha) \subseteq R\langle\alpha\rangle$.

Let P be a prime ideal of $R\langle\alpha\rangle$ and put $p = P \cap R$. Then $P \not\supseteq I_{[\alpha]}^{d}\varphi'_{\alpha}(\alpha)R\langle\alpha\rangle$ if and only if $R\langle\alpha\rangle_p$ is unramified over R_p. Indeed, if $P \supseteq I_{[\alpha]}^{d}\varphi'_{\alpha}(\alpha)R\langle\alpha\rangle$, then $R\langle\alpha\rangle \supseteq I_{[\alpha]}^{d-1}\varphi'_{\alpha}(\alpha)R\langle\alpha\rangle$, and so either $P \supseteq I_{[\alpha]}$ or $P \supseteq I_{[\alpha]}^{d-1}\varphi'_{\alpha}(\alpha)R\langle\alpha\rangle$. If $P \supseteq I_{[\alpha]}$, then $p \supseteq I_{[\alpha]}$. If $P \not\supseteq I_{[\alpha]}$, then $P \supseteq I_{[\alpha]}^{d-1}\varphi'_{\alpha}(\alpha)R\langle\alpha\rangle_p$ implies that $\varphi'_{\alpha}(\alpha)R\langle\alpha\rangle_p \subseteq PR\langle\alpha\rangle_p$ and that $\varphi'_{\alpha}(\alpha)R\langle\alpha\rangle_p \neq R\langle\alpha\rangle_p$. Hence $R\langle\alpha\rangle_p$ is ramified over R_p by Proposition 5.2.19. Conversely assume that $R\langle\alpha\rangle_p$ is unramified over R_p. Then $p \not\supseteq I_{[\alpha]}$ and $\varphi'_{\alpha}(\alpha)R\langle\alpha\rangle_p = R\langle\alpha\rangle_p$ by Proposition 5.2.19. So we have $P \not\supseteq I_{[\alpha]}^{d}\varphi'_{\alpha}(\alpha)R\langle\alpha\rangle_p$. ∎

5.3 Kernels of Derivations on Simple Extensions

Let $\mathbf{Z}$ denote the set of all integers and $\mathbf{Q}$ the field of the rational numbers. For an element $\alpha \in L$, put $A := R[\alpha]$. Let $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism of a polynomial ring $R[X]$ to $R[\alpha]$ sending X to α. Let $E^{(\alpha)} := \mathrm{Ker}(\pi)$. We denote by $\Omega_R(A)$ the module of differentials of A over R and by $d_{A/R}$ the universal R-derivation $A \to \Omega_R(A)$. Then $D := \mathrm{Ker}(d_{A/R})$ is an R-subalgebra of A. If the prime ring k of A is equal to $\mathbf{Z}/p\mathbf{Z}$ with p a positive prime number, then we have $a\alpha^p \in D$, and hence A is a finite D-module. Therefore D is Noetherian by Eakin-Nagata theorem (*cf.* [M2]). In the case $k = \mathbf{Z}$, D is not necessarily Noetherian. Let $\mathsf{c}(E^{(\alpha)})$ denote the ideal of R generated by the coefficients of the polynomials in $E^{(\alpha)}$. Then it is easy to see that $E^{(\alpha)} \subseteq \mathsf{c}(E^{(\alpha)})R[X]$ and $\mathsf{c}(E^{(\alpha)})A \cap R = \mathsf{c}(E^{(\alpha)})$. It is also easy to see that $E^{(\alpha)} \neq (0)$. Let $\widetilde{E}^{(\alpha)} := \{f'(X) \mid f(X) \in E^{(\alpha)}\}$, where $f'(X)$ denotes the derivative of $f(X)$ in the usual sense. Then it is easy to see that $\pi(\widetilde{E}^{(\alpha)})$ is an ideal of A and that the map $\delta : A \to A/\pi(\widetilde{E}^{(\alpha)})$ defined by $\delta(g(\alpha)) = g'(\alpha) \ mod\ \pi(E^{(\alpha)})$ is a well-defined R-derivation. In this case, there exists an A-module isomorphism $\rho : A/\pi(E^{(\alpha)}) \to \Omega_R(A)$ such that $\rho\delta = d_{A/R}$ (*cf.* [AK]). Hence we have $\mathrm{Ker}(d_{A/R}) = \mathrm{Ker}(\delta)$. Note that, for an element $f(X) \in R[X]$, there exists an element $F(X) \in R[X]$ such that $F'(X) = f(X)$ and $F(0) = 0$. We then set $\int_0^{\alpha} f(X)dX = F(\alpha)$.

Lemma 5.3.1 *Let* $D := \mathrm{Ker}(d_{A/R})$. *Then* $D = R + \sum_{f(X)\in E^{(\alpha)}} \int_0^{\alpha} f(X)dX$.

Proof Let $f(X)$ be an element of $E^{(\alpha)}$ and let $a = \int_0^{\alpha} f(X)dX$. Then we have $\delta(a) = f(\alpha) \ mod\ \pi(\widetilde{E}^{(\alpha)}) = 0$, and hence $a \in D$. Conversely, let $g(\alpha)$ be an element of D. Then we have $g'(\alpha) \in \pi(E^{(\alpha)})$ and hence we can find an element $f(X)$ of $E^{(\alpha)}$ satisfying $g'(\alpha) = f'(\alpha)$. Let $h(X) = g'(X) - f'(X)$ and $c = g(0) - f(0)$. Then $h(X) \in E^{(\alpha)}$ and $c \in R$. Since $g(\alpha) - f(\alpha) = c + \int_0^{\alpha} h(X)dX$

and $f(\alpha) = 0$, we have $g(\alpha) = c + \int_0^\alpha h(X)dX$, which completes the proof. $\square$

Lemma 5.3.2 *Let a be a nonzero element of $\mathsf{c}(E^{(\alpha)})$. Then there exists a positive integer r such that $a\alpha^{r+i} \in D + D\alpha + \cdots + D\alpha^{r-1}$ for every $i \geq 0$.*

Proof As is seen, there exists an element $f(X) = a_0X^n + a_1X^{n-1} + \cdots + a_n$ of $E^{(\alpha)}$ such that $a = a_j$ for some $0 \leq j \leq n$. Let $s = n(n+1)/2$. We show that $r = s + n + 1 - j$ satisfies the required condition. Let $\gamma_i - \int_0^\alpha X^i f(X)dX$ for each $i \geq 0$. Then each γ_i is an element of D by Lemma 5.3.1. Moreover, we have

$$\begin{pmatrix} \gamma_i \\ \gamma_{i+1} \\ \vdots \\ \gamma_{i+n} \end{pmatrix} = M_i \begin{pmatrix} a_0\alpha^{i+n+1} \\ a_1\alpha^{i+n} \\ \vdots \\ a_n\alpha^{i+1} \end{pmatrix}$$

for every $i \geq 0$, where

$$M_i = \begin{pmatrix} \frac{1}{i+n+1} & \frac{1}{i+n} & \cdots & \frac{1}{i+1} \\ \frac{\alpha}{i+n+2} & \frac{\alpha}{i+n+1} & \cdots & \frac{\alpha}{i+2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\alpha^n}{i+2n+1} & \frac{\alpha^n}{i+2n} & \cdots & \frac{\alpha^n}{i+n+1} \end{pmatrix}$$

Let

$$m_i = \det \begin{pmatrix} \frac{1}{i+n+1} & \frac{1}{i+n} & \cdots & \frac{1}{i+1} \\ \frac{1}{i+n+2} & \frac{1}{i+n+1} & \cdots & \frac{1}{i+2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{i+2n+1} & \frac{1}{i+2n} & \cdots & \frac{1}{i+n+1} \end{pmatrix}$$

Then we can check that m_i is a nonzero rational number (*cf.* [OnSuY2]) and $\det M_i = m_i\alpha^s$. Hence, putting L_i be the cofactor matrix of M_i, we have

$$m_i^{-1}L_i \begin{pmatrix} \gamma_i \\ \gamma_{i+1} \\ \vdots \\ \gamma_{i+n} \end{pmatrix} = \alpha^s \begin{pmatrix} a_0\alpha^{i+n+1} \\ a_1\alpha^{i+n} \\ \vdots \\ a_n\alpha^{i+1} \end{pmatrix}$$

Note that every component of $m_i^{-1}L_i$ belongs to $\mathbf{Q} + \mathbf{Q}\alpha + \cdots + \mathbf{Q}\alpha^s$. Since each γ_k is an element of D and $r \geq s+1$, it follows that

$$a_j\alpha^{r+i} \in D + D\alpha + \cdots + D\alpha^s \subseteq D + D\alpha + \cdots + D\alpha^{r-1}$$

for every $i \geq 0$. This completes the proof. ∎

Lemma 5.3.3 *The element α is integral over D if and only if $\mathsf{c}(E^{(\alpha)}) = R$.*

Proof If $\mathsf{c}(E^{(\alpha)}) = R$, then it follows from Lemma 5.3.2 that there exists a positive integer r such that $\alpha^r \in D + D\alpha + \cdots + D\alpha^{r-1}$. Thus α is integral over D. Conversely, assume that α is integral over D. Then we have

$$\alpha^r + c_1\alpha^{r-1} + \cdots + c_r = 0 \tag{1}$$

for some positive integer r and $c_1, \ldots, c_r \in D$. Suppose that $\mathsf{c}(E^{(\alpha)}) \neq R$. Let $\psi : A \to A/\mathsf{c}(E^{(\alpha)})A$ be the canonical homomorphism. Then we have $A/\mathsf{c}(E^{(\alpha)})A \cong (R[X]/E^{(\alpha)})/(\mathsf{c}(E^{(\alpha)})R[X]/E^{(\alpha)}) \cong R[X]/\mathsf{c}(E^{(\alpha)})R[X] \cong (R/\mathsf{c}(E^{(\alpha)}))[X]$, which yields that $\psi(\alpha)$ is algebraically independent over $R/\mathsf{c}(E^{(\alpha)})$. On the other hand, it follows by relation (1) that

$$\psi(\alpha)^r + \psi(c_1)\psi(\alpha)^{r-1} + \cdots + \psi(c_r) = 0 \tag{2}$$

Note that we have $\psi(\alpha) \in R/\mathsf{c}(E^{(\alpha)})$ for every $c \in D$ since $D \subseteq R + \mathsf{c}(E^{(\alpha)})A$ by Lemma 5.3.1. Then relation (2) shows that $\psi(c)$ is not algebraically independent over $R/\mathsf{c}(E^{(\alpha)})$, which is a contradiction. ∎

We are now ready to state and prove the following theorem.

Theorem 5.3.4 *Assume that R contains the rational number field $\boldsymbol{Q}$ and let $A := R[\alpha]$. Let $D := \mathrm{Ker}(d_{A/R})$. Then the following three conditions are equivalent:*

(1) *D is finitely generated R-algebra;*

(2) *D is Noetherian;*

(3) $\mathsf{c}(E^{(\alpha)}) = R$.

Proof We have only to check that (3) implies (1) and that (2) implies (3). Suppose that $\mathsf{c}(E^{(\alpha)}) = R$. Then α is integral over D by Lemma 5.3.3, and so

$$\alpha^r + c_1\alpha^{r-1} + \cdots + c_r = 0$$

for some positive integer r and $c_1, \ldots, c_r \in D$. Put $B := R[c_1, \ldots, c_r]$. Then A is a finite D-module. Thus D is finitely generated over R. Next, suppose that D

is Noetherian and let $a \in \mathsf{c}(E^{(\alpha)})$ be a nonzero element of A. Then there exists a positive integer r such that

$$a\alpha^{r+i} \in M := D + D\alpha + \cdots + C\alpha^{r-1}$$

for every $i \geq 0$ by Lemma 5.3.2. Since a is a nonzero element of A, this implies that $D[\alpha^r] \subseteq (1/a)M$. Hence $D[\alpha^r]$ is a finite D-module because D is Noetherian and $(1/a)M$ is a finite D-module. Thus α^r is integral over D, and so is α over D. So we have $\mathsf{c}(E^{(\alpha)}) = R$ by Lemma 5.3.3. ∎

Corollary 5.3.5 *Assume that R contains $\boldsymbol{Q}$. Let $A := R[\alpha]$. If A is flat over R, then $D := \mathrm{Ker}(d_{A/R})$ is finitely generated over R.*

Proof Since $\mathsf{c}(E^{(\alpha)}) \neq (0)$, if A is flat over R, then we have $\mathsf{c}(E^{(\alpha)}) = R$ (*cf.* [OR, Corollary 1.1.3]). So the assertion follows from Theorem 5.3.4. ∎

The converse of Corollary 5.3.5 seems not to hold in general (*cf.* [OnSuY2, Example 2.3]). However, the converse holds if the extension $R \subseteq A = R[\alpha]$ is anti-integral. In fact, when $A := R[\alpha]$ is anti-integral over R, A is flat over R if and only if $\mathsf{c}(E^{(\alpha)}) = J_{[\alpha]} = R$ by Proposition 2.5.4. Hence we have the following theorem.

Theorem 5.3.6 *Assume that $\alpha \in L$ an anti-integral element of degree d over R. Let $A := R[\alpha]$ and let $D := \mathrm{Ker}(d_{A/R})$. If R contains $\boldsymbol{Q}$, then the following three conditions are equivalent:*

(1) *D is finitely generated R-algebra;*

(2) *D is Noetherian;*

(3) *A is flat over R.*

Remark 5.3.7 Let $R \subseteq A := R[\alpha]$ be an anti-integral extension of degree d and assume that R contains $\mathbf{Q}$. Then $I_{[\alpha]}\varphi_\alpha(X)R[X] = \mathrm{Ker}(\pi)$, and hence, putting $\beta_i := \int_0^\alpha X^i\varphi_\alpha(X)dX$ for each $i \geq 0$, we have $D = R + I_{[\alpha]}\beta_0 + I_{[\alpha]}\beta_1 + \cdots$ by Lemma 5.3.1. ∎

Example 5.3.8 Let R be a polynomial ring $k[x, y]$ over a field $k := \mathbf{Q}$, and let $\alpha := y/x$. Then $\varphi_\alpha(X) = X - (y/x)$ and $I_{[\alpha]} = xR$. It is easy to see that the extension $R \subseteq A := R[\alpha]$ is anti-integral. Since

$$\beta_i = \int_0^\alpha X^i(X - \frac{y}{x})dX = -\frac{y^{i+2}}{(i+1)(i+2)x^{i+2}}$$

we have $I_{[\alpha]}\beta_i = y^{i+2}/x^{i+1}R$ for every $i \geq 0$. It follows from Remark 5.3.7 that

$$D = k[x, y, y^2/x, y^3/x^2, \dots]$$

which is not Noetherian. Note that $J_{[\alpha]} = \mathsf{c}(I_{[\alpha]}\varphi_\alpha(X))R \neq R$, i.e., A is not flat over R by Theorem 2.3.6.

Chapter 6

The Unit-Groups of Extensions

Let R be a Noetherian integral domain with the quotient field K and L be an algebraic extension field of K such that the degree $[L : K]$ of L over K equals to d. Let α be a nonzero element of L. We assume that $L = K(\alpha)$; that is, L is the quotient field of $R[\alpha]$. We denote the unit group of R by $U(R)$; that is,

$$U(R) := \{a \in R \mid ab = ba = 1 \text{ for some element } b \in R\}$$

In this chapter, we consider the relations among $U(R[\alpha])$, $U(R[\alpha]) \cap R$, and the subgroup of $U(R[\alpha])$ generated by the semigroup $U(R[\alpha]) \cap R$, in the case of an anti-integral extension $R[\alpha]$ of R. Let $I_\beta := \{r \in R \mid r\beta \in R\} = R :_R \beta$ for any $\beta \in L$ and $I_\beta^S := \{r \in S \mid r\beta \in S\} = S :_S \beta$ for a ring S such that $R \subseteq S \subseteq L$. Note that $I_\beta = I_\beta^R$. The natural numbers, the ring of integers, and a polynomial ring over R are written as $\mathbf{N}$, $\mathbf{Z}$, and $R[X]$, respectively.

When $R[\alpha]$ is a super-primitive and LCM-stable extension over R, we shall give a criterion whether $\beta \in U(R[\alpha])$ or not for an element β of $R[\alpha]$.

Let P be a prime ideal of height 1 in $R[\alpha]$. Set $p := P \cap R$. Since $R[\alpha]$ is LCM-stable over R, it is known that p is a prime ideal of height 1 in R from [SY2].

It is easy to see that $\beta \in U(R[\alpha])$ if and only if $\beta^{-1} \in R_p[\alpha]$ for all $p \in \mathrm{Dp}_1(R)$, where $\mathrm{Dp}_1(R)$ denotes the set of all prime ideals of R of depth 1. So we may assume that (R, m) is a local domain of depth 1.

First, we shall give a criterion whether $\beta \in U(R[\alpha])$ or not in the case that α is integral over R and $R[\alpha]$ is a free R-module of rank d. In short, we call this case an "integral case."

Next, in the general case, replacing α by a suitable element of $R[\alpha]$, we may assume that α satisfies that $R[\alpha^{-1}] \subseteq R[\alpha] = R[\alpha]$, and moreover we get that $(\alpha^{-1})^\ell \beta \in R[\alpha^{-1}]$ and $(\alpha^{-1})^\ell \beta$ is integral over R for a suitable integer $\ell > 0$. So the general case is reduced to an integral case.

Throughout this chapter, we use the following notation.

Let R be a Noetherian integral domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic field extension L of the quotient field K of R and $\pi : R[X] \longrightarrow R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Let $I_{[\alpha]} := \bigcap_{i=1}^d (R :_R \eta_i)$. For $f(X) \in R[X]$, let $\mathsf{c}(f(X))$ denote the ideal generated by the coefficients of

$f(X)$, that is, the content ideal of $f(X)$. Let $J_{[\alpha]} := I_{[\alpha]}\mathsf{c}(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called an *anti-integral* element of degree d over R if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, $R[\alpha]$ is called an *anti-integral extension* of R. The element α is called a *super-primitive* element of degree d over R if $J_{[\alpha]} \not\subseteq p$ for all primes p of depth 1. It is known that a super-primitive element is an anti-integral element (Theorem 2.2.8). (See Chapters 2 and 3 for details.)

6.1 The Unit-Groups of Anti-Integral Extensions

For $\alpha \in L$, the set $\{a \in R | \sqrt{aR} \supseteq I_{[\alpha]}$ and $J_{[\alpha]} + aR = R\}$ and $U(R[\alpha]) \cap R$ are semigroups, where $\sqrt{aR}$ denotes a radical ideal of aR.

When $R[\alpha]$ is an anti-integral extension of R and $\alpha \in K$, we proved that

$$U(R[\alpha]) \cap R = \{a \in R | \sqrt{aR} \supseteq I_\alpha \quad \text{and} \quad J_{[\alpha]} + aR = R\}$$

in [KYJ, Theorem 9]. We obtain the same result as above for any algebraic element $\alpha \in L$.

Theorem 6.1.1 *Let $\alpha \in L$ be an anti-integral element of degree d over R. Then $U(R[\alpha]) \cap R = \{a \in R | \sqrt{aR} \supseteq I_{[\alpha]}$ and $J_{[\alpha]} + aR = R\}$.*

Proof ($\Rightarrow$) Take $a \in U(R[\alpha]) \cap R$ and let $p_1, \ldots, p_n$ be the minimal prime divisors of aR. Since α is anti-integral over R, we see that $\Delta_{R[\alpha]/R} = V(\tilde{J}_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}}$ by Theorem 4.3.2, where $\Delta_{R[\alpha]/R} := \{p \in \mathrm{Spec}(R) | pR[\alpha] = R[\alpha]\}$ and $\Gamma_{J_{[\alpha]}} := \{p \in \mathrm{Spec}(R) | p + J_{[\alpha]} = R\}$. Since $aR[\alpha] = R[\alpha]$, it follows that $p_i R[\alpha] = R[\alpha]$ and hence $p_i \in \Delta_{R[\alpha]/R}$. Thus $p_i \supseteq \tilde{J}_{[\alpha]} \supseteq I_{[\alpha]}$ and $p_i + J_{[\alpha]} = R$ for all i, which means that $\sqrt{aR} = p_1 \cap \cdots \cap p_n \supseteq I_{[\alpha]}$ and $J_{[\alpha]} + \sqrt{aR} = J_{[\alpha]} + (p_1 \cap \cdots \cap p_n) = R$. It is easy to see that $J_{[\alpha]} + \sqrt{aR} = R$ if and only if $J_{[\alpha]} + aR = R$.
($\Leftarrow$) Take $a \in R$ satisfying $\sqrt{aR} \supseteq J_{[\alpha]}$ and $J_{[\alpha]} + aR = R$. Suppose that $a \notin U(R[\alpha]) \cap R$. Then $aR[\alpha] \neq R[\alpha]$. Let $P_1, \ldots, P\ell$ be the minimal prime divisors of $aR[\alpha]$. Put $p_i = P_i \cap R$. Then $p_i R[\alpha] \neq R[\alpha]$ for some i $(1 \leq i \leq \ell)$ because $\sqrt{aR[\alpha]} \neq R[\alpha]$. Put $p := p_i$. We have $I_{[\alpha]} \subseteq \sqrt{aR} \subseteq p$, but $J_{[\alpha]} + aR \subseteq p \neq R$, which is a contradiction. Thus $a \in U(R[\alpha]) \cap R$. $\square$

We give the following theorem.

Theorem 6.1.2 *Assume that α is an anti-integral element of degree d over R and R satisfies the altitude formula. If $\sqrt{I_{[\alpha]}} = p_1 \cap p_2 \cap \cdots \cap p_n$ ($I_{[\alpha]} \neq R$, and $p_i \in \mathrm{Spec}(R)$ for each i), then $\mathrm{ht}(p_i) = 1$ for each i ($1 \leq i \leq n$).*

First, we prove the following lemma.

Lemma 6.1.3 *If α is an anti-integral element of degree d over R, then α is an anti-integral element of degree d over R_p for each $p \in Spec(R)$.*

Proof Consider an exact sequence: $0 \longrightarrow I \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0$. Since α is anti-integral over R, I is generated by some polynomials of degree d over R. From the exact sequence

$$0 \longrightarrow I_p \longrightarrow R_p[X] \longrightarrow R_p[\alpha] \longrightarrow 0$$

we have that I_p is generated by some polynomials of degree d over R_p. Hence α is anti-integral of degree d over R_p. ∎

[The proof of Theorem 6.1.2] Fix i and put $p = p_i$. Since p is a minimal prime divisor of $I_{[\alpha]}$, it follows that $(I_{[\alpha]})_p$ is pR_p-primary. Suppose that $\mathrm{ht}(p) > 1$. Since R_p satisfies the altitude formula, we have that $\mathrm{ht}((I_{[\alpha]})_p) > 1$, noting that $(I_{[\alpha]})_p = \bigcap_i (R_p :_{R_p} \eta_i)$. We recall that $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Also, α is integral over R_p by Lemma 6.1.4. By Lemma 6.1.3, we have that α is an anti-integral element of degree d over R_p. Therefore $R_p[\alpha]$ is a free R_p-module of rank d by Corollary 2.3.5 and Theorem 2.3.8. Since α is integral over R_p, it follows that $\eta_i \in R_p$ for each i. Hence we have

$$(I_{[\alpha]})_p = \bigcap_i (I_{\eta_i})_p = \bigcap_i (R_p :_{R_p} \eta_i) = R_p$$

which is a contradiction. Thus we conclude $\mathrm{ht}(p) = 1$. ∎

Lemma 6.1.4 *Assume that R satisfies the altitude formula and $\mathrm{ht}(I_{[\alpha]}) > 1$. If α is anti-integral of degree d over R, then it is integral over R.*

Proof Let $\widetilde{R}$ denote the integral closure of R in L. Note that $\widetilde{R}$ is a Krull domain (*cf.* [M2]). Let P be any prime ideal of height 1 in $\widetilde{R}$ and put $p := P \cap R$. Since R satisfies the altitude formula, it follows that p is also of height 1. Since $\mathrm{ht}(I_{[\alpha]}) > 1$, we have that p does not contain $I_{[\alpha]}$. Therefore $\mathrm{ht}(I_{\eta_i}) > 1$ and so p does not contain I_{η_i} for each i. Thus we have $\eta_i \in R_p$ and so $\varphi_\alpha(X) \in R_p[X]$. Hence $\varphi_\alpha(\alpha) = 0$ yields that α is integral over R_p. From the fact that $\widetilde{R} = \cap \widetilde{R}_P$, we see that $\alpha \in \widetilde{R}$. ∎

Corollary 6.1.5 *We assume that R satisfies the altitude formula (dimension formula) and that α is anti-integral of degree d over R. If* $\mathrm{ht}(I_{[\alpha]}) > 1$, *then $R[\alpha]$ is a free R-module of rank d, i.e., $R[\alpha] = R \oplus R\alpha \oplus \cdots \oplus R\alpha^{d-1}$.*

Remark 6.1.6 In Corollary 6.1.5, if $\sqrt{aR} \supseteq I_{[\alpha]}$, then $a \in U(R)$. So we have $U(R[\alpha]) \cap R = U(R)$. ∎

Theorem 6.1.7 *Let α be a nonzero element of L. If $R[\alpha]$ is an anti-integral extension of R and a flat R-module, then $U(R[\alpha]) \cap R = \{a \in R | \sqrt{aR} \supseteq I_{[\alpha]}\}$.*

Proof Take $a \in R$ such that $\sqrt{aR} \supseteq I_{[\alpha]}$. Then $\sqrt{aR} + J_{[\alpha]} \supseteq J_{[\alpha]}$. Since $R[\alpha]$ is flat over R, we have that $J_{[\alpha]} = R$ by Theorem 2.3.6 and so $\sqrt{aR} + J_{[\alpha]} = R$. Hence $aR + J_{[\alpha]} = R$. Thus we have that $a \in U(R[\alpha]) \cap R$ by Theorem 6.1.1. The converse has been proved already. ∎

Remark 6.1.8 We assume that $x \in R$ and that $1/x$ is anti-integral over R. Then the following conditions are equivalent:

(1) $a \in U(R[1/x]) \cap R$;

(2) $\sqrt{aR} \supseteq xR$;

(3) There exists an element $b \in R$ such that $x^n = ab$ for some natural number n. ∎

Let $R \subseteq A$ be Noetherian integral domains. We consider the subgroup of $U(A)$ generated by the semigroup $U(A) \cap R$.

Definition 6.1.9 We denote by S^0 the subgroup of $U(A)$ generated by a subset $S \subseteq U(A)$.

Next we consider some examples.

Example 6.1.10 (An example such that $U(A)/U(R)$ is not a torsion-free group.) Let k be a field. Put $R = k[X, Y, (Y/X)^2, (X/Y)^2]$ and $A = R[1/X]$, where X, Y are indeterminates. Then $A = k[X, Y, 1/X, 1/Y]$. We have that $U(R) = U(k)$, $U(A)/U(R) \cong \mathbf{Z} \oplus \mathbf{Z}$, and $U(A) \cap R/U(R) \cong \mathbf{N} \oplus \mathbf{N}$. Hence $(U(A) \cap R)^0 = U(A)$. Also, Y/X is a torsion-element in $U(A)/U(R)$.

Example 6.1.11 Let k be a field and X, Y are indeterminates. Put $R = k[X, Y, X/Y]$ and $A = R[1/X]$. Then

$$A = k[X, Y, 1/X, 1/Y]$$

and

$$U(R) = U(k)$$
$$U(A)/U(R) \cong \mathbf{Z} \oplus \mathbf{Z}$$

and

$$U(A) \cap R/U(R) \cong \mathbf{N} \oplus \mathbf{N}$$

Thus $(U(A) \cap R)^0 = U(A)$.

Remark 6.1.12 Let $x \in R, x \neq 0$. Then we have $(U(R[1/x]) \cap R)^0 = U(R[1/x])$. ∎

Theorem 6.1.13 *We assume that R is a Noetherian local domain containing an infinite field k and $p \in \mathrm{Spec}(R)$ is of height 1 at most. Also, we assume that p is not a radical ideal of a principal ideal in R. Let $\alpha \in L$ be anti-integral over R. If $\sqrt{I_\alpha} = p$ and $J_{[\alpha]} = R$, that is, $R[\alpha]$ is a flat R-module, then $U(R[\alpha]) \cap R = U(R)$ and $U(R[\alpha]) \neq U(R)$.*

Proof Since R is local and $J_{[\alpha]} = R$, we may assume that $a_0\alpha^d + a_1\alpha^{d-1} + \cdots + a_d = 0$ and $a_d = 1$ where each $a_i \in R$. Since $\alpha(a_0\alpha^{d-1} + \cdots + a_{d-1}) = -1$, we have that $\alpha \in U(R[\alpha])$ and α is not contained in $U(R)$. If $a \in U(R[\alpha]) \cap R$ such that a is not contained in $U(R)$, then $\sqrt{aR} \supseteq I_{[\alpha]}$ and $aR + J_{[\alpha]} = R$ by Theorem 6.1.1. From the fact that $\sqrt{aR} \supseteq \sqrt{I_{[\alpha]}} = p$, we have $p = \sqrt{aR}$. This contradicts the assumption. Thus $U(R[\alpha]) \cap R = U(R)$. ∎

6.2 Invertible Elements of Super-Primitive Ring Extensions

We start with the following lemma.

Lemma 6.2.1 *Let R be a ring and let $R[\alpha] = R + R\alpha + \cdots + R\alpha^{d-1}$ be a ring extension of R. Then every element $\beta \in R[\alpha]$ has an integral relation of degree d over R, that is, there exists a monic polynomial $f(X)$ of degree d over R such that $f(\beta) = 0$.*

Proof Writing $\beta\alpha^{i-1} = a_{i1} + a_{i2}\alpha + \cdots + a_{id}\alpha^{d-1}$, where $a_{ij} \in R$, we have

$$\beta \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{d-1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{dd} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{d-1} \end{pmatrix}$$

Therefore we get that $\det(a_{ij} - \delta_{ij}\beta) = 0$. By expansion of this determinant, we obtain that β has an integral relation of degree d over R. $\square$

We recall the following definitions (Definition 2.2.1):

(i) $\alpha \in L$ is called an anti-integral element of degree d over R if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element, we say that $R[\alpha]$ is an anti- integral extension of R.

(ii) $\alpha \in L$ is called a super-primitive element of degree d over R if $J_{[\alpha]} \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R)$. When α is a super-primitive element, we say that $R[\alpha]$ is a super-primitive extension of R.

Using the result of Lemmas 6.2.1, we have the following proposition.

Proposition 6.2.2 *Let $R[\alpha]$ be a free R-module of rank d, that is, $R[\alpha] = R + R\alpha + \cdots + R\alpha^{d-1}$. Let $\beta \in R[\alpha]$ be an element satisfying $\beta^d + a_1\beta^{d-1} + \cdots + a_d = 0$, where all $a_i \in R$. Assume that $[K(\beta) : K] = d$. Then $\beta \in U(R[\alpha])$ if and only if $a_d \in U(R)$.*

Proof ($\Leftarrow$) Writing $\beta(\beta^{d-1} + a_1\beta^{d-2} + \cdots + a_{d-1}) = -a_d \in U(R)$, we have that $\beta \in U(R[\alpha])$.
($\Rightarrow$) Assume that $\beta \in U(R[\alpha])$. Since β^{-1} has a monic relation of degree d from Lemma 6.2.1, we conclude that $J_{[\beta^{-1}]} = R$ and so β^{-1} is a super-primitive element. Since a super-primitive element is an anti-integral element (Theorem 2.2.8), we conclude that $R[\beta^{-1}]$ is integral over R by Theorem 2.3.2. So β^{-1} has a monic relation of degree d, which yields that $a_d \in U(R)$. $\square$

Using the result of Proposition 6.2.2, we shall have a criterion whether $\beta \in U(R[\alpha])$ or not, where β is an integral or anti-integral element of $R[\alpha]$.

The following theorem plays an important role in this section.

Theorem 6.2.3 *Assume that α is an integral and anti-integral element of degree d over R. For $\beta \in R[\alpha]$, we assume that $[K(\beta) : K] = d$. Write $\beta\alpha^i = \sum_{j=0}^{d-1} a_{i+1\,j+1}\alpha^j$ where $i = 0, 1, \ldots, d-1$. Then $\beta \in U(R[\alpha])$ if and only if $\det(a_{ij}) \in U(R)$.*

Proof By the assumption, it follows that $R[\alpha] = R[\alpha] = R + R\alpha + \cdots + R\alpha^{d-1}$ is a free R-module of rank d. Now, $\det(a_{ij} - \delta_{ij}\beta) = 0$ is a monic relation of degree d with respect to β and the constant term of this equation of β is $\det(a_{ij})$. From Proposition 6.2.2, we complete the proof. $\square$

Theorem 6.2.4 *Assume that α is a super-primitive element of degree d over R and* grade $(I_{[\alpha]} + \eta_d I_{[\alpha]}) > 1$. *If $\beta \in R\langle\alpha\rangle$ and $[K(\beta) : K] = d$, then β has a monic relation of degree d over K. Therefore β is also a super-primitive element of degree d over R.*

Proof There exists a monic relation $\varphi_\beta(\beta) = 0$, where $\varphi_\beta(X) = X^d + \lambda_1 X^{d-1} + \cdots + \lambda_d \in K[X]$. We claim that all $\lambda_i \in R$. Suppose that $p \not\supseteq I_{[\alpha]}$ for $p \in \mathrm{Dp}_1(R)$. Then we have that $R_p[\alpha]$ is a free R_p-module of rank d from Theorem 2.2.5 and so $\beta \in R_p[\alpha] = R_p + R_p\alpha + \cdots + R_p\alpha^{d-1}$. Therefore β has a monic relation of degree d over R_p. This monic relation equals to $\varphi_\beta(\beta) = 0$. Hence $\lambda_i \in R_p$. Next, we assume that $p \supseteq I_{[\alpha]}$ for $p \in \mathrm{Dp}_1(R)$. Since grade $(I_{[\alpha]} + \eta_d I_{[\alpha]}) > 1$, we obtain that $p \not\supseteq \eta_d I_{[\alpha]} = I_{[\alpha^{-1}]}$. Hence $\beta \in R_p[\alpha^{-1}] = R_p + R_p(\alpha^{-1}) + \cdots + R_p(\alpha^{-1})^{d-1}$. $\lambda_i \in R_p$ is proved along the same lines as the case of $p \not\supseteq I_{[\alpha]}$. Hence $\lambda_i \in \bigcap_{p \in \mathrm{Dp}_1(R)} R_p = R (1 \le i \le d)$. Consequently, we proved the claim. Hence $I_{[\beta]} = R$ and so β is a super-primitive element of degree d over R by definition. ∎

Assume that $R \subseteq R[\alpha^{-1}] \subseteq R[\alpha]$. Then, for $\beta \in R[\alpha]$, there exists an integer $\ell > 0$ such that $(\alpha^{-1})^\ell \beta \in R[\alpha^{-1}]$. Since $\alpha^{-1} \in R[\alpha]$, it follows that $\alpha \in U(R[\alpha])$. Consequently, $\beta \in U(R[\alpha])$ if and only if $(\alpha^{-1})^\ell \beta \in U(R[\alpha])$. Since $R[\alpha^{-1}] \subseteq R[\alpha]$, we have that $R[\alpha^{-1}]$ is integral over R and so $(\alpha^{-1})^\ell \beta$ is an integral element over R. Hence we give a criterion whether $(\alpha^{-1})^\ell \beta \in U(R[\alpha])$ or not from Theorem 6.2.3.

Now, we shall consider the general case. Assume that R contains an infinite field k and assume that α is a super-primitive element of degree d over R and $R[\alpha]$ is LCM-stable over R. For $\beta \in R[\alpha]$, $\beta \in U(R[\alpha])$ if and only if $\beta \in U(R[\alpha]_p)$ for all $p \in \mathrm{Dp}_1(R)$. In case that $I_{[\alpha]} \not\subseteq p$, since a ring extension $R[\alpha]_p/R_p$ is integral, we give a criterion from Theorem 6.2.3.

Next, in case that $I_{[\alpha]} \subseteq p$, since α is a super-primitive element, we see that grade$(J_{[\alpha]}) > 1$. Replacing α by $\alpha - \mu$ for some $\mu \in k$, we may assume that $R_p \subseteq R_p[\alpha^{-1}] \subseteq R_p[\alpha]$. Hence we have reduced to the case that $R_p \subseteq R_p[\alpha^{-1}] \subseteq R_p[\alpha]$.

Summing up the above statements, we obtain the following remark.

Remark 6.2.5 Assume that R contains an infinite field k and $R[\alpha]$ is a super-primitive extension of R. Also, assume that $R[\alpha]$ is LCM-stable over R. Then, for $\beta \in R[\alpha]$,

(1) $\beta \in U(R[\alpha])$ if and only if $\beta \in U(R[\alpha]_p)$ for all $p \in \mathrm{Dp}_1(R)$.

(2) If $I_{[\alpha]} \not\subseteq p$, then we give a criterion whether $\beta \in U(R[\alpha]_p)$ or not, using Theorem 6.2.3. If $I_{[\alpha]} \subseteq p$, then replacing α by a suitable element, we give a remark whether $\beta \in U(R[\alpha]_p)$ or not, using Theorem 6.2.3. ∎

Remark 6.2.6 If $\mathrm{grade}(I_\alpha + \eta_d I_\alpha) > 1$, then the later case (2) in Remark 6.2.5 is the following:
We give a criterion whether $\alpha \in U(R_p)$ or not. Indeed, if $p \in \mathrm{Dp}_1(R)$, then $p \not\supseteq I_\alpha$ or $p \not\supseteq \eta_d I_\alpha$. If $p \not\supseteq I_\alpha$, then α is integral over R_p. If $p \supseteq I_\alpha$ and $p \not\supseteq \eta_d I_\alpha$, then $R_p \subseteq R_p[\alpha^{-1}] \subseteq R_p[\alpha]$. ∎

Now we shall consider the problem along the same lines as the preceding argument for a general ring extension. Let A be a ring extension of R. Assume that A is not necessarily a simple ring extension of R but L is an algebraic extension of K, where L is the quotient field of A.

Theorem 6.2.7 *Let A be a ring extension of R. Suppose that A is a finite R-module of rank n and A_p is a flat R_p-module for all $p \in \mathrm{Dp}_1(R)$. For $\beta \in A$, assume that $[K(\beta) : K] = n$ and $\varphi_\beta(X) = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n$ is the monic minimal polynomial of β over K. Then $\beta \in U(A)$ if and only if $\lambda_1, \ldots, \lambda_n \in R$, $\lambda_n \in U(R)$. (Thus β is an anti-integral element in this case.)*

Proof ($\Leftarrow$) Writing $\beta^{-1} = -\lambda_n^{-1}(\beta^{n-1} + \lambda_1\beta^{n-2} + \cdots + \lambda_{n-1}) \in A$, we have that $\beta \in U(A)$.
($\Rightarrow$) $A_p = R_p\alpha_1 + \cdots + R_p\alpha_n$ for all $p \in \mathrm{Dp}_1(R)$. Therefore $\beta\alpha_i = \sum_j a_{ij}\alpha_j$, where $a_{ij} \in R_p$. Hence β has a monic relation of degree n and so $\varphi_\beta(\beta) = 0$. Hence $\lambda_i \in R_p$ for all $p \in \mathrm{Dp}_1(R)$. Thus $\lambda_i \in R$. Since $\beta^{-1} \in A$, we get that $\lambda_n \in U(R)$. ∎

Especially, when R is a normal domain, we have the following result.

Theorem 6.2.8 *Let R be a Noetherian normal domain and let A be a ring extension of R such that $[L : K] = n$. Also, let $\beta \in A$, β is integral over R, and $[K(\beta) : K] = n$. Let $\varphi_\beta(X) = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n$ $(\lambda_i \in K)$ be the monic minimal polynomial of β over K. Then $\beta \in U(A)$ if and only if $\lambda_1, \ldots, \lambda_n \in R$ and $\lambda_n \in U(R)$.*

Proof ($\Leftarrow$) Writing $\beta^{-1} = -\lambda_n^{-1}(\beta^{n-1} + \lambda_1\beta^{n-2} + \cdots + \lambda_{n-1}) \in A$, we see that $\beta \in U(A)$.
($\Rightarrow$) Let B be the integral closure of R in A. Then $\beta \in B$ and so $\lambda_1, \ldots, \lambda_n \in R$ and $\lambda_n \in U(R)$ from Theorem 6.2.7. ∎

For $\beta \in A$, if A is integral over R, then we get a powerful criterion whether $\beta \in U(A)$ or not by Theorem 6.2.3.

Consequently, we come to a problem when a ring extension A/R is an integral extension. On this problem, we prove the following theorem in the

case that L is a finite separable algebraic extension of K, where L is the quotient field of A.

Theorem 6.2.9 *Let A be an integral domain with the quotient field L and a ring extension of R. Assume that L/K is a finite separable algebraic extension. If A_p is flat and integral over R_p for all $p \in \mathrm{Dp}_1(R)$, then A is an integral extension over R.*

Proof Since L/K is a finite separable algebraic extension, L has only a finite number of subfields containing K.

Case (1).There exists $\beta \in A$ such that $[K(\beta) : K] = [L : K] = n$. ($L$ is a simple field extension over K.) We are done in this case.

Case (2). $[K(\gamma) : K] < n$ for all $\gamma \in A$. In this case, there exists an element $\beta \in A_p$ such that $[K(\beta) : K] = n$ and $[K(\beta + \gamma) : K] = n$. Consequently, if β is integral over R for all β such that $[K(\beta) : K] = n$, then A is integral over R. Let $[K(\beta) : K] = n$ and $\varphi_\beta(X) = X^n + \eta_1 X^{n-1} + \cdots + \eta_n$ ($\eta_i \in K$) be the monic minimal polynomial of β over K. Since A_p is a free R_p-module of rank n for all $p \in \mathrm{Dp}_1(R)$, we get that $A_p = R_p\eta_1 + R_p\eta_2 + \cdots + R_p\eta_n$. Therefore $\beta\eta_i = \sum_j a_{ij}\eta_i$ where $a_{ij} \in R_p$ and so $\varphi_\beta(X) = \det(\delta_{ij}X - a_{ij})$. Consequently, $\eta_i \in R_p$ for all $p \in \mathrm{Dp}_1(R)$. Hence $\varphi_\beta(X) \in R[X]$. Thus we have that A is integral over R. $\square$

Remark 6.2.10 If R is a normal domain, we get that A_p is a flat R_p-module for all $p \in \mathrm{Dp}_1(R)$ because R_p is a DVR. $\square$

Chapter 7

Exclusive Extensions of Noetherian Domains

In this chapter, we use the following notations used before unless otherwise specified:

R: a Noetherian integral domain,

$K := K(R)$: the quotient field of R,

L: an algebraic field extension of K,

α: a nonzero element of L,

$d = [K(\alpha) : K]$,

$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$, the minimal polynomial of α over K,

$I_{[\alpha]} := \bigcap_{i=1}^{d}(R :_R \eta_i)$,

$I_a := R :_R aR$ for $a \in K$,

$\tilde{I}_{[\alpha]} := \bigcap_{i=1}^{d-1}(R :_R \eta_i)$.

It is clear that for $a \in K$, $I_{[a]} = I_a$ from definitions.

$$J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d),$$
$$\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}).$$

Note that $I_{[\alpha]}$, $\tilde{I}_{[\alpha]}$, $J_{[\alpha]}$, and $\tilde{J}_{[\alpha]}$ are ideals of R. We also use the following standard notation:

$$\mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) | \mathrm{depth}(R_p) = 1\}$$

Recall the following notation:
Let $\pi : R[X] \longrightarrow R[\alpha]$ be the R-algebra homomorphism sending X to α. The element α is called an *anti-integral* element of degree d over R if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, $R[\alpha]$ is called an *anti-integral extension* of R. (See Chapter 3 for details.) For $f(X) \in R[X]$, let $\mathrm{c}(f(X))$ denote the ideal generated by the coefficients of $f(X)$, that is, the

content ideal of $f(X)$. Let $J_{[\alpha]} := I_{[\alpha]}\mathrm{c}(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called a *super-primitive* element of degree d over R if $J_{[\alpha]} \not\subseteq p$ for all primes p of depth 1. It is known that a super-primitive element is an anti-integral element (see Theorem 2.2.8 for details).

7.1 Subring $R[\alpha] \cap K$ of Anti-Integral Extensions

We examine the following problem:

Problem. Let R be a Noetherian integral domain, K denote the quotient field of R, and α be a super-primitive element over R of degree d. What is a subring $R[\alpha] \cap K$?

Especially we intend to study when $R[\alpha] \cap K = R$ holds and to investigate what is $R[\alpha] \cap K$ if $R[\alpha] \cap K \neq R$.

We start with the following result.

Proposition 7.1.1 *Assume that $\alpha \in L$ is anti-integral over R of degree d. Then*

$$R[\alpha] \cap K \subseteq \bigcap R_p \ (p \in \mathrm{Dp}_1(R) \textit{ with } I_{[\alpha]} \not\subseteq p)$$

Proof It suffices to show that $R_p \supseteq R_p[\alpha] \cap K$ for any $p \in \mathrm{Dp}_1(R)$ with $p \not\supseteq I_{[\alpha]}$. Since $(I_{[\alpha]})_p = R_p$, $R_p[\alpha]$ is integral over R_p, i.e., α is integral over R_p (*cf.* Theorem 2.3.2). Hence $R_p[\alpha]$ is a finitely generated free R_p-module and we have an expression: $R_p[\alpha] = R_p + R_p\alpha + \cdots + R_p\alpha^{d-1}$ (a direct sum). We see that $R_p[\alpha] \cap K$ is contained in the first term R_p on the right-hand side. So we have our conclusion. $\square$

Lemma 7.1.2 *Assume that α is anti-integral over R. For any $\beta \in R[\alpha] \cap K$, there exists an integer $n > 0$ such that $I_\beta \supseteq (I_{[\alpha]})^n$.*

Proof Since $\beta \in K$, we have $I_{[\beta]} = I_\beta := R :_R \beta$ and so I_β is a divisorial ideal of R. Then any prime divisor p of I_β is of depth 1. To show $I_\beta \supseteq (I_{[\alpha]})^n$ for some n, we have only to show that $I_{[\alpha]} \subseteq p$ for each prime divisor of I_β. Suppose that $p \not\supseteq I_{[\alpha]}$. Then $\beta \in R[\alpha] \cap K \subseteq R_p$ by Proposition 7.1.1 and hence $I_\beta \not\subseteq p$, which contradicts the assumption $I_\beta \subseteq p$. $\square$

Lemma 7.1.3 *Assume that α is anti-integral over R. For any $\beta \in R[\alpha] \cap K$, there exists $a \in R$ such that $(\beta - a)I_{[\beta]} \subseteq I_{[\alpha]}\eta_d$.*

Proof Let $\beta = a_0 + a_1\alpha + \cdots + a_n\alpha^n$ with some $a_i \in R$. Then $(a_0 - \beta) + a_1\alpha + \cdots + a_n\alpha^n = 0$. For any $b \in I_{[\beta]}$, we have $b(a_0 - \beta) + ba_1X + \cdots + ba_nX^n \in E^{(\alpha)} = I_{[\alpha]}\varphi_\alpha(X)R[X]$, where the last equality results from the anti-integrality of α. Considering the constant term, we have $b(a_0 - \beta) \in I_{[\alpha]}\eta_d$. Thus $(\beta - a_0)I_{[\beta]} \subseteq I_{[\alpha]}\eta_d$. The element a is a_0, which we are looking for. ∎

Lemma 7.1.4 *Assume that α is anti-integral over R and that $R[\alpha] \cap K \neq R$.*

(1) *There exists $\beta \in R[\alpha] \cap K \setminus R$ such that $I_{[\alpha]} \subseteq I_{[\beta]}$.*

(2) *If $\gamma \in R[\alpha] \cap K \setminus \{0\}$, then $I_{\gamma - a} = I_\gamma$ for any $a \in R$.*

Proof The second assertion (2) is obvious by definition.
To prove (1), take $\beta \in R[\alpha] \cap K \setminus R$. Then there exists an integer $n > 0$ such that $(I_{[\alpha]})^n \subseteq I_{[\beta]} = I_\beta$, i.e., $(I_{[\alpha]})^n\beta \subseteq R$, by Lemma 7.1.2. Take a minimal n having the above property. If $a\beta \in R$ for all $a \in (I_{[\alpha]})^{n-1}$, then $(I_{[\alpha]})^{n-1}\beta \subseteq R$, that is, $(I_{[\alpha]})^{n-1} \subseteq I_\beta = I_{[\beta]}$, which contradicts the minimality of n. Hence there exists $b \in (I_{[\alpha]})^{n-1}$ such that $b\beta \notin R$ and $I_{[\alpha]}b\beta \subseteq (I_{[\alpha]})^n\beta \subseteq R$. By the choice of b, we have $b\beta \in R[\alpha] \cap K \setminus R$. So we have only to take $b\beta$ as a required element β. ∎

Let J be a fractional ideal of R and let $\mathcal{R}(J) := J :_K J$, which is an over-ring of R.

Remark 7.1.5 It is known that α is anti-integral over R and $\mathcal{R}(I_{[\alpha]}) = R$ if and only if α is super-primitive over R (see Theorem 2.3.10). ∎

Theorem 7.1.6 *Assume that α is a super-primitive element over R and that $\eta_d \in R$. Then $R[\alpha] \cap K = R$.*

Proof Suppose that $R[\alpha] \cap K \neq R$. Then there exists an element $\beta \in R[\alpha] \cap K$ with $\beta \notin R$ such that $I_{[\alpha]} \subseteq I_{[\beta]}$ by Lemma 7.1.4. Replace β by $\beta - a$ for some $a \in R$, we can assume that $\beta I_{[\beta]} \subseteq I_{[\alpha]}\eta_d$ by Lemma 7.1.3. Recall that $I_{[\beta]} = I_{[\beta - a]}$ for any $a \in R$ by Lemma 7.1.4. Since $\eta_d \in R$, we have $I_{[\alpha]}\eta_d \subseteq I_{[\alpha]}$ and so $\beta I_{[\alpha]} \subseteq \beta I_{[\beta]} \subseteq I_{[\alpha]}\eta_d \subseteq I_{[\alpha]}$, which means that $\beta \in \mathcal{R}(I_{[\alpha]}) = R$, a contradiction. Therefore $R[\alpha] \cap K = R$. ∎

Example 7.1.7 Let k be a field and s, t denote indeterminates. Let $R = k[s, t]$ be a polynomial ring. Let α be an element which has the minimal polynomial $\varphi_\alpha(X) = X^2 + (t/s)X + 1$. Then $\eta_d = 1 \in R$, $I_{[\alpha]} = sR$, and $J_{[\alpha]} = (s, t)R$.

Hence α is a super-primitive element over R. So we have $R[\alpha] \cap K = R$ by Theorem 7.1.6. Note that $R[\alpha]$ is not integral over R.

A strong version of Theorem 7.1.6 is given by the following:

Theorem 7.1.8 *Assume that α is super-primitive over R. Then*

$$R[\alpha] \cap K \subseteq \bigcap_p R_p \quad (p \in \mathrm{Dp}_1(R) \textit{ with } p \not\supseteq I_{\eta_d})$$

Proof Take $p \in \mathrm{Dp}_1(R)$ with $p \not\supseteq I_{\eta_d}$. Then $\eta_d \in R_p$ and $\mathcal{R}((I_{[\alpha]})_p) = R_p$ by Remark 7.1.5. So $R_p[\alpha] \cap K = R_p$ by Theorem 7.1.6. Thus $R[\alpha] \cap K \subseteq R_p$ and therefore $R[\alpha] \cap K \subseteq \bigcap R_p$ $(p \in \mathrm{Dp}_1(R)$ with $p \not\supseteq I_{\eta_d})$. ∎

For $a \in R$, we have $\varphi_\alpha(a) = a^d + \eta_1 a^{d-1} + \cdots + \eta_d \in K$. Let M_α denote the ideal $\sum_{a \in R} I_{\varphi_\alpha(a)}$ of R, where $I_{\varphi_\alpha(a)} = R :_R \varphi_\alpha(a)$.

Theorem 7.1.9 *Assume that α is a super-primitive element over R. If $M_\alpha \not\subseteq p$ for any $p \in \mathrm{Dp}_1(R)$, then $R[\alpha] \cap K = R$.*

Proof The assumption $M_\alpha \not\subseteq p$ for any $p \in \mathrm{Dp}_1(R)$ yields that there exists $I_{\varphi_\alpha(a)} \not\subseteq p$ for each $p \in \mathrm{Dp}_1(R)$. Note that $\varphi_\alpha(a)$ is the constant term of $\varphi_\alpha(X + a) = \varphi_{\alpha - a}(X)$ and that $\alpha - a$ is super-primitive over R (*cf.* Proposition 2.2.10). Hence $R[\alpha]_p \cap K = R[\alpha - a]_p \cap K \subseteq R_p$ by Theorem 7.1.6. Thus $R[\alpha] \cap K \subseteq R[\alpha]_p \cap K \subseteq R_p$ for all $p \in \mathrm{Dp}_1(R)$. So we have $R[\alpha] \cap K \subseteq \bigcap_{p \in \mathrm{Dp}_1(R)} R_p = R$. ∎

Now we are devoted to investigating what is $R[\alpha] \cap K$ in the case it is not equal to R. In the next proposition, we remark that α is not necessarily assumed to be anti-integral over R.

Proposition 7.1.10 *If $\eta_1, \ldots, \eta_{d-1} \in R$, then $R[\alpha] \cap K = R[\eta_d]$.*

Proof Since $\eta_d = -(\alpha^d + \eta_1 \alpha^{d-1} + \cdots + \eta_{d-1}\alpha) \in R[\alpha] \cap K$, we have $R[\eta_d] \subseteq R[\alpha] \cap K$. Note that $\varphi_\alpha(X)$ is a minimal polynomial of α over K and $\eta_d \in K$. It follows that $R[\eta_d][\alpha] = R[\eta_d] + R[\eta_d]\alpha + \cdots + R[\eta_d]\alpha^{d-1}$ (a direct sum). Considering $R[\eta_d] \subseteq K[\alpha] = K + K\alpha + \cdots + K\alpha^{d-1}$, we conclude that $R[\alpha] \cap K \subseteq R[\eta_d][\alpha] \cap K = R[\eta_d]$. Hence $R[\alpha] \cap K = R[\eta_d]$. ∎

Example 7.1.11 Let R be the same as in Example 7.1.7. Let α be an element which has the minimal polynomial $\varphi_\alpha(X) = X^2 + X + (1/s)$. Then $I_{[\alpha]} =$

$M_\alpha = sR$, $J_{[\alpha]} = R$, and $R[\alpha] \cap K = R[1/s]$ by Proposition 7.1.10. Note here that α is not super-primitive over R.

The example above is generalized as follows:

Proposition 7.1.12 *Assume that α is super-primitive over R. If $\tilde{I}_{[\alpha]} + I_{\eta_d} = R$, then $R[\alpha] \cap K = R[\tilde{I}_{[\alpha]}\eta_d]$.*

Proof Put $D = R[\tilde{I}_{[\alpha]}\eta_d]$. Then it is obvious that $R[\alpha] \cap K \supseteq D$. Take $P \in \mathrm{Spec}(D)$ with $\mathrm{depth}(D_P) = 1$, and put $p = P \cap R$. Since $\tilde{I}_{[\alpha]} + I_{\eta_d} = R$, it follows that $p \not\supseteq I_{\eta_d}$ or $p \not\supseteq \tilde{I}_{[\alpha]}$. If $p \not\supseteq I_{\eta_d}$, then $\eta_d \in R_p$, and hence $R[\alpha] \cap K \subseteq R_p[\alpha] \cap K = R_p \subseteq D_P$ by Theorem 7.1.6. If $p \not\supseteq \tilde{I}_{[\alpha]}$, then $\eta_1, \ldots, \eta_{d-1} \in R_p$. So by Proposition 7.1.10, we have $R[\alpha] \cap K \subseteq R_p[\alpha] \cap K = R_p[\eta_d] \subseteq D_P$. In any case, we have $R[\alpha] \cap K \subseteq \bigcap_{P \in \mathrm{Dp}_1(D)} D_P = D = R[\tilde{I}_{[\alpha]}\eta_d]$. This completes the proof. ∎

The proof of Proposition 7.1.12 also shows the following.

Corollary 7.1.13 *Assume that α is super-primitive over R. If $P \cap R \not\supseteq I_{\eta_d} + \tilde{I}_{[\alpha]}$ for every $P \in \mathrm{Dp}_1(R[\tilde{I}_{[\alpha]}\eta_d])$, then $R[\alpha] \cap K = R[\tilde{I}_{[\alpha]}\eta_d]$.*

In the next two propositions, we show that α anti-integral (respectively super-primitive) over R is still anti-integral (respectively super-primitive) over a Noetherian intermediate ring B in the case B is flat over R.

Let B be an intermediate ring between R and K. Put $I^B_{\eta_i} = B :_B \eta_i$, $I^B_{[\alpha]} = \bigcap_{i=1}^d I^B_{\eta_i}$, $J^B_{[\alpha]} = I^B_{[\alpha]}\mathfrak{c}(\varphi_\alpha(X))$ and $M^B_\alpha = \sum_{b \in B} I^B_{\varphi_\alpha(b)}$. If B is flat over R, then $I^B_{\eta_i} = I_{\eta_i}B$, $I^B_{[\alpha]} = I_{[\alpha]}B$, $J^B_{[\alpha]} = J_{[\alpha]}B$ and $M^B_\alpha = M_\alpha B$.

Theorem 7.1.14 *Assume that α is anti-integral over R, that $R \subseteq B \subseteq K$ and that B is a Noetherian ring and is flat over R. Then α is anti-integral over B.*

Proof Consider the following exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow R[X] \xrightarrow{\pi} R[\alpha] \longrightarrow 0$$

Tensoring it with B over R, we have the exact sequence of B-modules:

$$0 \longrightarrow \mathrm{Ker}(\pi) \otimes_R B \longrightarrow B[X] \xrightarrow{\pi^B} B[\alpha] \longrightarrow 0$$

Since α is anti-integral over R, we have $E^{(\alpha)} := \mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$ and $E_B^{(\alpha)} := \mathrm{Ker}(\pi^B)$. Since B is flat over R, $E_B^{(\alpha)} = E^{(\alpha)} \otimes_R B = I_{[\alpha]}\varphi_\alpha(X)$ $R[X] \otimes_R B = I_{[\alpha]}\varphi_\alpha(X)B[X]$. So we have the exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\pi^B)\varphi_\alpha(X)B[X] \longrightarrow B[X] \xrightarrow{\pi^B} B[\alpha] \longrightarrow 0$$

This shows that $E_B^{(\alpha)}$ is generated by polynomials of degree d in $B[X]$. Thus α is anti-integral over B by Proposition 2.2.3. $\square$

Note that if $R \subseteq B \subseteq K$ and B is a Noetherian ring and is flat over R, then $I_{[\alpha]}\varphi_\alpha(X)B[X] = I_{[\alpha]}^B\varphi_\alpha(X)B[X] = \mathrm{Ker}(\pi^B)$, where $\pi^B : B[X] \longrightarrow B[\alpha]$ is the canonical B-algebra homomorphism sending X to α.

Theorem 7.1.15 *Assume that α is super-primitive over R, that $R \subseteq B \subseteq K$, and that B is a Noetherian ring and is flat over R. Then α is also super-primitive over B.*

Proof Since B is flat over R, we have $I_{\eta_i}^B = I_{\eta_i}B$ for all $i = 1, \ldots, d$. Thus we have:

$$\begin{aligned} J_{[\alpha]}^B &= I_{[\alpha]}^B \mathsf{c}(\varphi_\alpha(X)) \\ &= (\bigcap_{i=1}^{d} I_{\eta_i}^B)(1, \eta_1, \ldots, \eta_d) \\ &= (\bigcap_{i=1}^{d} I_{\eta_i}B)(1, \eta_1, \ldots, \eta_d) \\ &= (\bigcap_{i=1}^{d} I_{\eta_i})(1, \eta_1, \ldots, \eta_d)B \\ &= J_{[\alpha]}B \end{aligned}$$

Since α is super-primitive over R, $J_{[\alpha]} \not\subseteq p$ for every $p \in \mathrm{Dp}_1(R)$. Since B is flat over R, $J_{[\alpha]}B \not\subseteq P$ for every $P \in \mathrm{Dp}_1(B)$. Thus $J_{[\alpha]}^B = J_{[\alpha]}B \not\subseteq P$ for every $P \in \mathrm{Dp}_1(B)$, which means that α is super-primitive over B. $\square$

Corollary 7.1.16 *Let R, α, and B be the same as in Theorem 7.1.15. Assume that $\eta_d \in B$. Then $R[\alpha] \cap K \subseteq B$.*

Proof Theorem 7.1.15 yields that α is super-primitive over B. Since $\eta_d \in B$, we have $B[\alpha] \cap K = B$ by Theorem 7.1.12, whence $R[\alpha] \cap K \subseteq B[\alpha] \cap K = B$. $\square$

Corollary 7.1.17 *Assume that α is super-primitive over R and that $R[\eta_d]$ is flat over R. Then $R[\alpha] \cap K \subseteq R[\eta_d]$.*

Proof Apply Corollary 7.1.16 to the case $B = R[\eta_d]$. $\square$

Proposition 7.1.18 *Let B be an intermediate Noetherian ring between R and K and assume that α is super-primitive over R and that $R[\alpha]$ is flat over R. Then α is super-primitive over B and $B[\alpha]$ is flat over B.*

Proof Since $R[\alpha]$ is flat over R, it follows that $J_{[\alpha]} = R$ by Theorem 2.3.6. So $J^B_{[\alpha]} \supseteq J_{[\alpha]}B = B$ and hence $J^B_{[\alpha]} = B$. Thus α is super-primitive over B. $B[\alpha]$ is flat over B by Theorem 2.3.6. $\square$

Proposition 7.1.19 *Let B be an intermediate Noetherian ring between R and $R[\alpha] \cap K$. Assume that α is super-primitive over B and that $M^B_\alpha \not\subseteq p$ for all $p \in \mathrm{Dp}_1(B)$. Then $R[\alpha] \cap K = B$.*

Proof Since α is super-primitive over B and $M^B_\alpha \not\subseteq p$ for all $p \in \mathrm{Dp}_1(B)$, we have $B \subseteq R[\alpha] \cap K \subseteq B[\alpha] \cap K = B$ by Theorem 7.1.9, and hence $R[\alpha] \cap K = B$. $\square$

Corollary 7.1.20 *Assume that α is super-primitive over R. Let $a_1, \ldots, a_n$ be elements in R such that $(a_1, \ldots, a_n)R \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R)$. Let $B_i := R[1/a_i]$ for $1 \leq i \leq n$. Assume that $M^{B_i}_\alpha \not\subseteq p$ for all $p \in \mathrm{Dp}_1(B_i)$ and all i. Then $R[\alpha] \cap K = R$.*

Proof Since B_i $(1 \leq i \leq n)$ is flat over R, α is super-primitive over B_i for all i by Theorem 7.1.15. So $R[\alpha] \cap K = B_i$ by Proposition 7.1.19. Hence $R[\alpha] \cap K \subseteq \bigcap_{i=1}^n B_i$. Since $(a_1, \ldots, a_n)R \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R)$, we have $\bigcap_{i=1}^n B_i \subseteq \bigcap_{p \in \mathrm{Dp}_1(R)} R_p = R$. Thus $R \subseteq R[\alpha] \cap K \subseteq R$ and hence $R[\alpha] \cap K = R$. $\square$

In the rest of this section, we consider the case of lower degree and some examples. We restrict ourselves to the case $d = 2$ and show some examples.

Proposition 7.1.21 *Let a, b be elements in R with $a \neq 0$ and let $\varphi_\alpha(X) = X^2 + (1/a)X + (b/a)$. Then $R[\alpha] \cap K = R$.*

Proof Since $J_{[\alpha]} = a(1, 1/a, b/a)R = R$, the element α is a super-primitive element over R. The constant term of $\varphi_{\alpha+b}(X)$ belongs to R. Hence $R[\alpha] \cap K = R$ by Theorem 7.1.6. ∎

Proposition 7.1.22 *Let $\varphi_\alpha(X) = X^2 + \eta_1 X + \eta_2$ with $\eta_i \in K$.*

(1) *Assume that $\eta_1 = b/a$ with a regular sequence $a, b \in R$ and that $I_{\eta_1} \subseteq I_{\eta_2}$. Then $R[\alpha] \cap K = R$.*

(2) *If $R[\alpha] \cap K = R$, then $I_{\eta_1} \subseteq I_{\eta_2}$.*

Proof (1) Since $I_{\eta_1} = aR$ and $I_{\eta_1} \subseteq I_{\eta_2}$, $\eta_2 = c/a$. Hence $\varphi_\alpha(X) = X^2 + (b/a)X + (c/a)$. In this case, $J_{[\alpha]} = (a, b, c)R$ and $J_{[\alpha]} \not\subseteq p$ for $p \in \mathrm{Dp}_1(R)$. So α is super-primitive over R. By Theorem 7.1.15, α is super-primitive over $R[1/a]$ and $R[1/b]$. Thus $R[\alpha] \cap K \subseteq R[1/a][\alpha] \cap K = R[1/a]$. On the other hand, we can apply Proposition 7.1.21, we have $R[\alpha] \cap K \subseteq R[1/b]$ and hence $R[\alpha] \cap K \subseteq R[1/a] \cap R[1/b] = R$ because a, b is a regular sequence. Therefore $R[\alpha] \cap K = R$.
(2) This follows from $\eta_2 I_{\eta_1} \subseteq R[\alpha] \cap K = R$. ∎

Example 7.1.23 Let R be a polynomial ring $k[s, t]$ over a field k. Let $\varphi_\alpha(X) = X^2 + (t/s)X + (1/s)$. Then by Proposition 7.1.22(1) $R[\alpha] \cap K = R$.

Example 7.1.24 Let R be the same as in Example 7.1.23. Let $\varphi_\alpha(X) = X^2 + (t/s)X + (1/s^2)$. It is clear that $1/s \in R[\alpha] \cap K$. Put $B = R[1/s]$. Then B is flat over R and α is super-primitive over R. Moreover $\eta_d = 1/s^2 \in B$. So $R[\alpha] \cap K \subseteq B = R[1/s]$ by Corollary 7.1.26. The inclusion $R[1/s] \subseteq R[\alpha] \cap K$ is obvious. Hence $R[\alpha] \cap K = R[1/s]$.

Proposition 7.1.25 *Let $\varphi_\alpha(X) = X^2 + (1/a)X + (c/b)$ with $a, b, c \in R$. Then $R[\alpha] \cap K = R[ac/b]$.*

Proof The inclusion $R[ac/b] \subseteq R[\alpha] \cap K$ is obvious. Put $u = ac/b$. Then $\varphi_\alpha(X) = X^2 + (1/a)X + (u/a)$. Hence $J_{[\alpha]}^{R[u]} = a(1, 1/a, u/a)R[u] = R[u]$, which means that α is super-primitive over $R[u]$. So we have $R[u][\alpha] \cap K = R[u]$ by Proposition 7.1.21. Thus $R[u] \subseteq R[\alpha] \cap K \subseteq R[u][\alpha] \cap K = R[u]$, which yields $R[\alpha] \cap K = R[u]$. ∎

Example 7.1.26 We use the same notations as in Example 7.1.23.

(i) Let $\varphi_\alpha(X) = X^2 + (1/s)X + (1/t)$. Then $R[\alpha] \cap K = R[s/t]$.

(ii) Let $\varphi_\alpha(X) = X^2 + (1/s)X + (1/t^2)$. Then $R[\alpha] \cap K = R[s/t^2]$.

7.2 Exclusive Extensions and Integral Extensions

Let R be a Noetherian integral domain and $R[X]$ a polynomial ring. Let γ be an element of a field extension L of the quotient field K of R. If $\gamma \in L$ is transcendental over R, then $R[\gamma] \cap K = R$, and if $\gamma \in K$ then $R[\gamma] \cap K = R[\gamma]$. So we have only to consider an element algebraic over R.

We study the following problem:

Problem. Let R be a Noetherian integral domain, K denote the quotient field of R, and α be an algebraic element over R of degree d. When does $R[\alpha] \cap K = R$?

Now we start with the following definition.

Definition 7.2.1 When $R[\alpha] \cap K = R$, we say that α is an *exclusive element* over R and that $R[\alpha]$ is an *exclusive extension* of R.

Proposition 7.2.2 *If $R[\alpha] \cap K = R$, then $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$.*

Proof Note that $\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0$. Take $a \in \tilde{I}_{[\alpha]}$. Then $a\eta_d = -(a\alpha^d + a\eta_1\alpha^{d-1} + \cdots + a\eta_{d-1}\alpha) \in R[\alpha] \cap K = R$. Thus $a \in I_{\eta_d}$, which means that $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$. ∎

Corollary 7.2.3 *Assume that α is super-primitive over R. When $\tilde{I}_{[\alpha]} + I_{\eta_d} = R$, the following statements are equivalent:*

(i) *α is exclusive over R, i.e., $R[\alpha] \cap K = R$,*

(ii) *$\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$.*

Proof (ii) $\Rightarrow$ (i): By Proposition 7.1.12, we have $R = R[\tilde{I}_{[\alpha]}\eta_d] = R[\alpha] \cap K$ because $\tilde{I}_{[\alpha]}\eta_d \subseteq R$.

(i) $\Rightarrow$ (ii) follows from Proposition 7.1.2. ∎

From Proposition 7.1.12, we have the following result:

Corollary 7.2.4 *If $I_{[\alpha]} = R$, then $R[\alpha] \cap K = R$. Moreover if α is super-primitive over R, then $I_{\eta_d} = R$, i.e., $\eta_d \in R$, implies $R[\alpha] \cap K = R$.*

The following lemma is very elementary, but we give a proof for convenience.

Lemma 7.2.5 *Let $f(X) \in R[X]$ be a monic polynomial and let $g(X) \in K[X]$. If $f(X)g(X)$ is a monic polynomial in $R[X]$, then $g(X) \in R[X]$.*

Proof Put $f(X) := X^n + a_1X^{n-1} + \cdots + a_n$ with $a_i \in R$ and $g(X) = \lambda_0X^m + \cdots + \lambda_m$ with $\lambda_i \in K$. Since $f(X)g(X)$ is monic, we have $\lambda_0 = 1$. Assume that $\lambda_0 = 1, \lambda_2, \ldots, \lambda_{\ell-1} \in R$ and $\lambda_\ell \notin R$. Then the coefficient of the degree $(\ell + n)$-term is $\lambda_\ell + a_1\lambda_{\ell-1} + \cdots + a_\ell\lambda_0 \in R$. Hence $\lambda_\ell \in R$, a contradiction. Thus $\lambda_i \in R$ for all i. So we have $g(X) \in R$. ∎

Recall the following, which has been proved in Section 2.3.

Proposition 7.2.6 *(cf. Theorem 2.3.2) Assume that R is normal. Then α is integral over R if and only if $I_{[\alpha]} = R$.*

Proof ($\Rightarrow$): Since R is normal, we have $\varphi_\alpha(X) \in R[X]$ by Theorem 2.3.2. By Theorem 2.2.9, α is super-primitive and hence anti-integral because R is a Krull domain. Hence $I_{[\alpha]} = R$ follows from Theorem 2.3.2.
($\Leftarrow$): Since $I_{[\alpha]} = R$, we have $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X] = \varphi_\alpha(X)R[X]$. Thus $\varphi_\alpha(\alpha) = 0$ gives rise to an integral dependence of α. So α is integral over R. ∎

Corollary 7.2.7 *Assume that R is normal. If α is integral over R, then α is an exclusive element over R.*

Proof We have $I_{[\alpha]} = R$ by Proposition 7.2.6 and it follows that $R[\alpha] \cap K = R$ by Corollary 7.2.6. ∎

Proposition 7.2.8 *If $\tilde{I}_{[\alpha]} = R$ and if α is exclusive over R, i.e., $R[\alpha] \cap K = R$, then $R[\alpha]$ is integral over R.*

Proof By Proposition 7.2.6, $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$ and hence $I_{[\alpha]} = \bigcap_{i=1}^{d} I_{\eta_i} = R$. Thus $\eta_1, \ldots, \eta_d \in R$, which means that $\varphi_\alpha(X) \in R[X]$ and $\varphi_\alpha(\alpha) = 0$. This yields that α is integral over R. ∎

In Lemma 1.1.3, we have seen that if $\alpha \in K$ is both integral and anti-integral over R then $\alpha \in R$. For the case $d \geq 0$, we have a similar result as follows.

Theorem 7.2.9 *If α is both anti-integral and integral over R, then α is exclusive over R, i.e., $R[\alpha] \cap K = R$.*

Proof Since α is integral over R, there exists a monic polynomial $f(X) \in R[X]$ such that $f(\alpha) = 0$. Since α is anti-integral over R, we have $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Hence $f(X) \in \mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. So there exists a

polynomial $g(X) \in I_{[\alpha]}R[X]$ such that $f(X) = \varphi_\alpha(X)g(X)$. Since both $f(X)$ and $\varphi_\alpha(X)$ are monic, $g(X)$ is monic. Thus $\varphi_\alpha(X) \in R[X]$ by Lemma 7.2.5, which shows that $I_{[\alpha]} = R$. Therefore $R[\alpha] \cap K = R$ by Corollary 7.2.4. $\square$

7.3 An Exclusive Extension Generated by a Super-Primitive Element

Let R be a Noetherian domain and K its quotient field. Let α be an algebraic element over K with the minimal polynomial $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. In Section 7.1, we have shown that if $R[\alpha] \cap K = R$ then $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$. Our objective of this section is to show the converse of this result under certain assumptions, which will be established in Theorem 7.3.7.

A nonzero element α in L is called *co-monic* if α^{-1} is integral over R and a polynomial $f(X) \in R[X]$ is *co-monic* if the constant term of $f(X)$ is 1.

Proposition 7.3.1 *Assume that α is anti-integral over R of degree d. Then the following conditions are equivalent:*

(i) *α is co-monic;*

(ii) *$\eta_d I_{[\alpha]} = R$;*

(iii) *there exists an element $a \in R$ such that $\eta_d = 1/a$ and $I_{[\alpha]} = aR$;*

(iv) *there exists a co-monic polynomial $g(X) \in R[X]$ of degree d with $g(\alpha) = 0$;*

(v) *$R[\alpha^{-1}]$ is a free R-module of rank d.*

Proof (i) $\Rightarrow$ (ii): Take a co-monic polynomial $f(X)$ in $R[X]$ satisfying $f(\alpha) = 0$. Then $f(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$ and hence $f(0) = 1 \in \eta_d I_{[\alpha]} \subseteq R$. So we have $\eta_d I_{[\alpha]} = R$.
(ii) $\Rightarrow$ (iii): There exists $a \in I_{[\alpha]}$ such that $\eta_d a = 1$. Hence $\eta_d = 1/a$ and $I_{[\alpha]} = aR$.
(iii) $\Rightarrow$ (iv): Since $I_{[\alpha]} = aR$, $g(X) := a\varphi_\alpha(X) \in R[X]$ is a required polynomial.
(iv) $\Rightarrow$ (v): The polynomial $f(Y) := X^{-d}g(X)$ in $R[Y]$ with $Y = 1/X$ is monic in Y of degree d and satisfies $f(\alpha^{-1}) = 0$. Hence $R[\alpha^{-1}]$ is a free R-module of rank d because α^{-1} is of degree d.
(v) $\Rightarrow$ (i) is obvious. $\square$

Recall the following result shown in Theorem 4.3.2:

Lemma 7.3.2 *Assume that α is an anti-integral element over R of degree d. Put $\Delta_{R[\alpha]/R} := \{p \in \mathrm{Spec}(R) \mid pR[\alpha] = R[\alpha]\}$ and $\Gamma_{J_{[\alpha]}} := \{p \in \mathrm{Spec}(R) \mid p + J_{[\alpha]}\} = R$. Then $\Delta_{R[\alpha]/R} = V(\tilde{J}_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}}$, where $V(\tilde{J}_{[\alpha]})$ denotes $\{p \in \mathrm{Spec}(R) \mid \tilde{J}_{[\alpha]} \subseteq p\}$.*

Remark 7.3.3 Under the assumptions in Lemma 7.3.2, if either $\tilde{J}_{[\alpha]} = R$ or $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ then $pR[\alpha] \neq R[\alpha]$ for all $p \in \mathrm{Dp}_1(R)$. ∎

Lemma 7.3.4 *Assume that α is a super-primitive element over R. The following statements are equivalent:*

(i) $\tilde{J}_{[\alpha]} = R$ *or* $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ *(i.e., $\tilde{J}_{[\alpha]}$ contains an R-regular sequence of length > 1; see [M2] for the definition);*

(ii) $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$.

Proof (i) $\Rightarrow$ (ii): First consider the case $\tilde{J}_{[\alpha]} \neq R$. Suppose that there exists $a \in \tilde{I}_{[\alpha]}$ but $a \notin I_{\eta_d}$. Then $\beta := a\eta_d \notin R$. Since $\beta I_{[\alpha]} = a(\eta_d I_{[\alpha]}) \subseteq R$ and $\beta\eta_i I_{[\alpha]} = (a\eta_i)\eta_d I_{[\alpha]} \subseteq R$ $(i = 1, \ldots, d-1)$, we have $\beta I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}) \subseteq R$. Since $\mathrm{grade}(\tilde{J}_{[\alpha]}) = \mathrm{grade}(I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})) > 1$, $\beta R_p = \beta(I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}))_p \subseteq R_p$ for all $p \in \mathrm{Dp}_1(R)$. Hence $\beta \in \bigcap_{\mathrm{Dp}_1(R)} R_p = R$, which is a contradiction. Thus $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$. Second, suppose that $\tilde{J}_{[\alpha]} = R$. By the assumption, there is an equation $1 = a_0 + a_1\eta_1 + \cdots + a_{d-1}\eta_{d-1}$ with $a_i \in I_{[\alpha]}$. So $\eta_d = a_0\eta_d + (a_1\eta_d)\eta_1 + \cdots + (a_{d-1}\eta_d)\eta_{d-1}$. Take $x \in \tilde{I}_{[\alpha]}$. Then $x\eta_d = xa_0\eta_d + (a_1\eta_d)x\eta_1 + \cdots + (a_{d-1}\eta_d)x\eta_{d-1} \in R$, and hence $x \in I_{\eta_d}$. Hence $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$.
(ii) $\Rightarrow$ (i): We may assume that $\tilde{J}_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}) \neq R$. Suppose that $\mathrm{grade}(\tilde{J}_{[\alpha]}) = \mathrm{grade}(I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})) = 1$. Then there exists $p \in \mathrm{Dp}_1(R)$ such that $I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}) \subseteq p$. Since $p \in \mathrm{Dp}_1(R)$, there exists $\beta \in K$ such that $\beta \notin R$ and $I_\beta = p$. Then $\beta I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}) \subseteq R$. Thus $(I_{[\alpha]}\beta)\eta_i \subseteq R$ for $i = 1, \ldots, d-1$ and $I_{[\alpha]}\beta \subseteq R$, which yield $I_{[\alpha]}\beta \subseteq \tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$. Hence $I_{[\alpha]}\beta\eta_d \subseteq R$, which shows that $J_{[\alpha]}\beta \subseteq R$. But since α is super-primitive over R, $J_{[\alpha]} \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R)$. Thus $\beta \in R$, which is a contradiction. So we conclude that $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$. ∎

Proposition 7.3.5 *Assume that α is super-primitive and co-monic over R. If $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$, then $R[\alpha] \cap K = R$, i.e., α is exclusive over R.*

Proof Suppose that $R[\alpha] \cap K \neq R$. Then there exists $\beta \in R[\alpha] \cap (K \setminus R)$. Write $\beta = c_0 + c_1\alpha + \cdots + c_n\alpha^n$ with $c_i \in R$. We shall show that $\alpha \in I_\beta R[\alpha]$. Since $\beta, c_0, \ldots, c_n \in K$ and α is an algebraic element of degree d over K, we have $n \geq d$. From this we have:

(1) $(\alpha^{-1})^n\beta = c_0(\alpha^{-1})^n + c_1(\alpha^{-1})^{n-1} + \cdots + c_n \in R[\alpha^{-1}]$.
Since α is co-monic, $R[\alpha^{-1}]$ is a free R-module of rank d by Proposition 7.3.1. So we rewrite (1) and have:

$$(\alpha^{-1})^n\beta = b_0(\alpha^{-1})^{d-1} + b_1(\alpha^{-1})^{d-2} + \cdots + b_{d-1}$$

with $b_i \in R$. Hence we obtain:
(2) $(\alpha^{-1})^n = \beta^{-1}b_0(\alpha^{-1})^{d-1} + \beta^{-1}b_1(\alpha^{-1})^{d-2} + \cdots + \beta^{-1}b_{d-1}$.
On the other hand, since $(\alpha^{-1})^n \in R[\alpha^{-1}]$, we have:
(3) $(\alpha^{-1})^n = a_0(\alpha^{-1})^{d-1} + a_1(\alpha^{-1})^{d-2} + \cdots + a_{d-1}$ with $a_i \in R$.
Comparing the coefficients of (2) with those of (3), we have $a_i = \beta^{-1}b_i$ $(0 \leq i \leq d-1)$, and hence $\beta a_i = b_i \in R$. Thus $(a_0, a_1, \ldots, a_{d-1})R \subseteq I_\beta$. It is obvious that $\alpha \in (a_0, \ldots, a_{d-1})R[\alpha] \subseteq I_\beta R[\alpha]$ by (3). Take $p \in \mathrm{Dp}_1(R)$ such that $I_\beta \subseteq p$. By (3), $\alpha^{-1} = a_0\alpha^{n-d} + a_1\alpha^{n-d+1} + \cdots + a_{d-1}\alpha^{n-1}$. So we get $\alpha^{-1} \in pR[\alpha]$. Thus $pR[\alpha]$ contains α, α^{-1}. So we have $pR[\alpha] = R[\alpha]$, which is a contradiction because of Remark 7.3.3 and Lemma 7.3.4. Therefore $R[\alpha] \cap K = R$. ∎

Remark 7.3.6 In Section 7.2, we showed that if α is super-primitive over R with $\eta_d \in R$ then α is exclusive.

In the next theorem, the implication (i) $\Rightarrow$ (ii) has been shown in Proposition 7.2.2 without any assumptions. ∎

Theorem 7.3.7 *Assume that R contains an infinite field k and that α is super-primitive over R. Then the following statements are equivalent:*

(i) *α is exclusive over R;*
(ii) $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$;
(iii) $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ *or* $\tilde{J}_{[\alpha]} = R$.

Proof The implication (i) $\Rightarrow$ (ii) was shown in Proposition 7.2.2.
(ii) $\Leftrightarrow$ (iii) follows from Lemma 7.3.4. So we have only to prove the implication (ii) $\Rightarrow$ (i). Take $p \in \mathrm{Dp}_1(R)$. If $I_{[\alpha]} \not\subseteq p$, then α is integral over R_p and hence $R[\alpha] \cap K \subseteq R_p[\alpha] \cap K = R_p$. Consider the case $I_{[\alpha]} \subseteq p$. Since α is super-primitive, $J_{[\alpha]} \not\subseteq p$. Since k is infinite, there exists $\lambda \in k$ such that $\alpha - \lambda$ is co-monic over R_p. Moreover $\alpha - \lambda$ is super-primitive by Proposition 2.2.10, Hence by Proposition 7.3.5, we have $R[\alpha] \cap K = R[\alpha - \lambda] \cap K \subseteq R_p[\alpha - \lambda] \cap K = R_p[\alpha] \cap K = R_p$. Thus $R[\alpha] \cap K \subseteq \bigcap_{p \in \mathrm{Dp}_1(R)} R_p = R$, and hence $R = R[\alpha] \cap K$, that is, $R[\alpha]$ is exclusive over R. ∎

Remark 7.3.8 Assume that R contains a field k. Let x be an indeterminate and put $S := R \otimes_k k(x)$. Then S contains the infinite field $k(x)$. Put $I^S_{[\alpha]} :=$

$\bigcap_{i=1}^{d}(S :_S \eta_i)$, $J_{[\alpha]}^S := I_{[\alpha]}^S(1, \eta_1, \ldots, \eta_d)S$ and $\tilde{J}_{[\alpha]}^S := I_{[\alpha]}^S(1, \eta_1, \ldots, \eta_{d-1})S$. Since S is faithfully flat over R, $I_{[\alpha]}^S = I_{[\alpha]}S$, $J_{[\alpha]}^S = J_{[\alpha]}S$, and $\tilde{J}_{[\alpha]}^S = \tilde{J}_{[\alpha]}S$. So if α is super-primitive (respectively anti-integral) over R, then so is α over S. Moreover α is exclusive over S if and only if so is α over R; $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$ if and only if $\tilde{I}_{[\alpha]}^S \subseteq I_{\eta_d}^S$; $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ if and only if $\mathrm{grade}(\tilde{J}_{[\alpha]}^S) > 1$; and α is exclusive over S, i.e., $S[\alpha] \cap K(x) = S[\alpha]$ if and only if α is exclusive over R, i.e., $R[\alpha] \cap K = R$. ∎

Lemma 7.3.9 (*cf.* Theorem 7.3.7) *Assume that α is super-primitive over R. Then the following statements are equivalent:*

(i) $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$;

(ii) $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ *or* $\tilde{J}_{[\alpha]} = R$.

Furthermore if R contains a field, then the following (iii) *is equivalent to* (i)*:*

(iii) *α is exclusive over R.*

Proof (i) $\Leftrightarrow$ (ii) follows from Lemma 7.3.4.
Next assume that R contains a field. We may assume that R contains an infinite field. Hence our conclusion (ii) $\Leftrightarrow$ (iii) follows from Theorem 7.3.7. ∎

Remark 7.3.10 Assume that R contains a field.

(i) When α is a super-primitive element over R, α is exclusive over R if and only if $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ by Lemma 7.3.9.

(ii) The equality $\tilde{J}_{[\alpha^{-1}]} = I_{[\alpha]}(\eta_1, \ldots, \eta_d)$ holds. This follows from the similar argument of Remark 2.5.14.

(iii) By Theorem 2.4.8, α is super-primitive over R if and only if so is α^{-1}. So by (i) and (ii) above, we have α^{-1} is exclusive over R if and only if $\mathrm{grade}(I_{[\alpha]}(\eta_1, \ldots, \eta_d)) > 1$.

(iv) If $\mathrm{grade}(I_{[\alpha]}(\eta_1, \ldots, \eta_{d-1})) > 1$, then both α and α^{-1} are exclusive over R by (i), (ii), and (iii). ∎

Proposition 7.3.11 *Assume that α is an anti-integral element of degree d over R. If $I_{[\alpha]}R[\alpha]_P = R[\alpha]_P$ for every $P \in \mathrm{Dp}_1(R[\alpha])$, then $R[\alpha] \cap K = R[\eta_1, \ldots, \eta_d]$.*

Proof It follows that $R[\alpha]_P \supseteq I_{[\alpha]}^{-1} I_{[\alpha]} R[\alpha]_P = I_{[\alpha]}^{-1} R[\alpha]_P \supseteq I_{[\alpha]}^{-1} \ni \eta_1, \ldots, \eta_d$ because $\bigcap_{P \in \mathrm{Dp}_1(R)} R[\alpha]_P = R[\alpha]$. Thus $R[\alpha] \cap K \supseteq R[\eta_1, \ldots, \eta_d] =: C$,

α is integral and anti-integral over C. Since $K(C) = K$ and $[K(\alpha) : K] = d$, $R[\alpha]$ is a free C-module $C + C\alpha + \cdots + C\alpha^{d-1}$. Hence $R[\alpha] \cap K = C$, as was to be shown. ∎

Theorem 7.3.12 *Assume that α is a super-primitive element of degree d over R and that R contains a field. Let $\phi : \mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R)$ be the canonical map induced from the inclusion $R \subseteq R[\alpha]$. Then the following statements are equivalent:*

(i) $\mathrm{Dp}_1(R)$ *is contained in the image of ϕ, i.e.,* $\mathrm{Im}(\phi) \supseteq \mathrm{Dp}_1(R)$*;*

(ii) *α is exclusive over R.*

Proof $\mathrm{Im}(\phi) \supseteq \mathrm{Dp}_1(R)$ if and only if $\tilde{J}_{[\alpha]p} = J_{[\alpha]p} = R_p$ for every $p \in \mathrm{Dp}_1(R)$ by Theorem 4.3.2. Since α is super-primitive over R, we see that $\mathrm{grade}(J_{[\alpha]}) > 1$. Hence we have $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$, that is, $\tilde{J}_{[\alpha]p} = R_p$ for every $p \in \mathrm{Dp}_1(R)$. So we conclude that α is exclusive over R by Lemma 7.3.9. Conversely, the set of diffusing points (i.e., $p \in \mathrm{Spec}(R)$ such that $pR[\alpha]_p = R[\alpha]_p$) is given by $\bigcap_{i=1}^{d-1} V(I_{[\alpha]}\eta_i) \setminus V(I_{[\alpha]}\eta_d) = V(\tilde{J}_{[\alpha]}) \setminus V(I_{[\alpha]}\eta_d)$ by Theorem 4.3.2. So take $p \in \mathrm{Dp}_1(R)$ such that $pR[\alpha]_p = R[\alpha]_p$. Then $p \supseteq \tilde{J}_{[\alpha]}$ and $p \not\supseteq I_{[\alpha]}\eta_d$. Thus we have $\mathrm{grade}(\tilde{J}_{[\alpha]}) = 1$, which yields that α is not exclusive over R by Lemma 7.3.9. Hence $qR[\alpha]_q \neq R[\alpha]_q$ for all $q \in \mathrm{Im}(\phi)$. ∎

Proposition 7.3.13 *Assume that $\alpha_1, \ldots, \alpha_n$ are super-primitive elements over R and that R contains a field. Put $A := R[\alpha_1, \ldots, \alpha_n]$ and let $\phi : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ be the canonical map induced from the inclusion $R \subseteq A$. If $\mathrm{Dp}_1(R)$ is contained in the image of ϕ, then each α_i $(1 \leq i \leq n)$ is exclusive over R.*

Proof Let $\psi_i : \mathrm{Spec}(A) \to \mathrm{Spec}(R[\alpha_i])$ and $\phi_i : \mathrm{Spec}(R[\alpha_i]) \to \mathrm{Spec}(R)$ be the canonical maps induced from the inclusion $R[\alpha_i] \subseteq A$ and $R \subseteq R[\alpha_i]$, respectively. Then $\phi = \phi_i \cdot \psi_i$ induces the inclusions $\mathrm{Im}(\phi_i) \supseteq \mathrm{Im}(\phi) \supseteq \mathrm{Dp}_1(R)$. Thus our conclusion follows Theorem 7.3.12. ∎

Problem. Is the converse statement of Proposition 7.3.13 valid?

Theorem 7.3.14 *Assume that α is a super-primitive element of degree d over R. If $\eta_d \in R$, then the canonical map $\phi : \mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R)$ is surjective.*

Proof Note that $\phi : \mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R)$ is surjective if and only if $V(\tilde{J}_{[\alpha]}) = V(J_{[\alpha]}) \subseteq V(I_{[\alpha]}\eta_d)$ (*cf.* Theorem 4.3.2). Since $\eta_d \in R$, we have $\eta_d I_{[\alpha]} \subseteq I_{[\alpha]}$, which implies that $V(\tilde{J}_{[\alpha]}) = V(J_{[\alpha]})$ by the construction of $J_{[\alpha]}$. So we have our conclusion. ∎

Note that $\phi : \mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R)$ is surjective and ϕ is flat if and only if ϕ is faithfully flat. So we obtain the following corollary.

Corollary 7.3.15 *Assume that $\eta_d \in R$. If $R[\alpha]$ is flat over R, then $R[\alpha]$ is faithfully flat over R.*

The following proposition gives rise to a condition for $\eta_d \in R$.

Proposition 7.3.16 *Assume that α is a super-primitive element of degree d over R and that R contains a field. Then the following statements are equivalent:*

(i) $\eta_d \in R$;

(ii) *α is exclusive over R and $I_{\eta_d} R[\alpha] = R[\alpha]$.*

Proof (i) $\Rightarrow$ (ii) follows from Lemma 7.3.9.
(ii) $\Rightarrow$ (i): Since $\eta_d \in I_{\eta_d}^{-1} R[\alpha] = I_{\eta_d}^{-1} I_{\eta_d} R[\alpha] \subseteq R[\alpha]$, we conclude that $\eta_d \in R[\alpha] \cap K = R$. ∎

Remark 7.3.17 Let I denote an ideal of R. Then

$$\mathrm{grade}(I) > 1 \;\Leftrightarrow\; I^{-1} = R$$

where $I^{-1} := R :_K I$. ∎

Theorem 7.3.18 *Assume that α is an anti-integral element of degree d over R. Let $f : \mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R)$ and $g : \mathrm{Spec}(R[\eta_d]) \to \mathrm{Spec}(R)$ be the canonical maps obtained from the inclusions $R \subseteq R[\alpha]$ and $R \subseteq R[\eta_d]$, respectively. If $\mathrm{grade}_{R[\alpha]}(J_{[\alpha]} R[\alpha]) > 1$, $\mathrm{grade}_{R[\alpha]}(J_{[\eta_d]} R[\alpha]) > 1$, and $\mathrm{grade}_{R[\eta_d]}(J_{[\alpha]} R[\eta_d]) > 1$, then $R[\alpha] \cap K = R[\eta_d]$ and $\mathrm{Im}(f) = \mathrm{Im}(g)$.*

Proof First note that $J_{[\alpha]} R[\eta_d] \subseteq J_{[\alpha]}^{R[\eta_d]}$ (here we use the notation as in Remark 7.3.8; put $S = R[\eta_d]$). So $\mathrm{grade}_{R[\eta_d]}(J_{[\alpha]} R[\eta_d]) > 1$ yields that $\mathrm{grade}_{R[\eta_d]}(J_{[\alpha]}^{R[\eta_d]}) > 1$, so that α is super-primitive over $R[\eta_d]$. Take $P \in \mathrm{Dp}_1(R[\alpha])$ and put $p := P \cap R$. Since $J_{[\alpha]} R[\alpha] \not\subseteq P$, we have either $J_{[\alpha]} \not\subseteq p$ or $I_{\eta_d} \not\subseteq p$. Thus $R_p[\alpha]$ is flat over R_p. So we have $I_{\eta_d} R_p[\alpha] = (R :_R \eta_d) R_p[\alpha] = R_p[\alpha] :_{R_p[\alpha]} \eta_d = R_p[\alpha]$. Thus $\eta_d \in R_p[\alpha]$ by Proposition 2.5.11. Therefore $\eta_d \in \bigcap_{P \in \mathrm{Dp}_1(R[\alpha])} R[\alpha]_P = R[\alpha]$. As mentioned above, α is super-primitive over $R[\eta_d]$. Since $\eta_d \in R[\eta_d]$, α is exclusive over $R[\eta_d]$. Hence the canonical map $\psi : \mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R[\eta_d])$ is surjective by Proposition 7.3.14.

Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(R[\eta_d][\alpha]) & = & \mathrm{Spec}(R[\alpha]) \\ \psi\downarrow & & g\downarrow \\ \mathrm{Spec}(R[\eta_d]) & \xrightarrow{f} & \mathrm{Spec}(R) \end{array}$$

Here we use that $\eta_d \in R[\alpha]$. Since ψ is surjective and $f \cdot \psi = g$, we conclude that $\mathrm{Im}(f) = \mathrm{Im}(g)$. ∎

We say that α is an *ultra-primitive element* of degree d over R if $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$, where $\overline{R}$ denotes the integral closure of R in K and $\mathcal{C}(\overline{R}/R)$ denotes the conductor between R and $\overline{R}$.

Proposition 7.3.19 *Assume that α is an ultra-primitive element of degree d. If* $\mathrm{grade}(I_{[\alpha]} :_R I_{[\alpha^{-1}]}) > 1$, *then $\eta_d \in R$.*

Proof Take $p \in \mathrm{Dp}_1(R)$. Then either $I_{[\alpha]} \not\subseteq p$ or $\mathcal{C}(\overline{R}/R) \not\subseteq p$. If $I_{[\alpha]} \not\subseteq p$, then $I_{[\alpha]} \subseteq I_{\eta_d} \not\subseteq p$, that is, $\eta_d \in R_p$. If $I_{[\alpha]} \subseteq p$, then $\mathcal{C}(\overline{R}/R) \not\subseteq p$ and hence R_p is a normal domain. Note that $I_{[\alpha^{-1}]p} = \eta_d I_{[\alpha]p} \subseteq I_{[\alpha]p}$ (*cf.* Lemma 2.4.6). The latter inclusion $\eta_d I_{[\alpha]p} \subseteq I_{[\alpha]p}$, that is, $\eta_d \in (I_{[\alpha]} :_K I_{[\alpha]})_p$ implies that η_d is integral over R_p, noting that $I_{[\alpha]p}$ is finitely generated over R_p. So we have $\eta_d \in R_p$. Therefore $\eta_d \in \bigcap_{p \in \mathrm{Dp}_1(R)} R_p = R$. ∎

Note that $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ with $\eta_j \in K$, where $\eta_0 := 1$, and put $\varphi_{\alpha^{-1}}(X) = X^d + \eta'_1 X^{d-1} + \cdots + \eta'_d$ with $\eta'_j \in K$, where $\eta'_0 := 1$. Then $\eta'_j = \eta_{d-j}/\eta_d$ for every j $(0 \le j \le d)$.

Put $\tilde{J}^{(i)}_{[\alpha]} := I_{[\alpha]}(\eta_0, \eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_d)$ for $(0 \le i \le d)$, and put $\tilde{J}^{(i)}_{[\alpha^{-1}]} := I_{[\alpha^{-1}]}(\eta'_0, \eta'_1, \ldots, \eta'_{i-1}, \eta'_{i+1}, \ldots, \eta'_d)$ for $(0 \le i \le d)$. It is clear that $\tilde{J}^{(d)}_{[\alpha]} = \tilde{J}_{[\alpha]}$ and $\tilde{J}^{(d)}_{[\alpha^{-1}]} = \tilde{J}_{[\alpha^{-1}]}$ by definition.

We start with the following lemma.

Lemma 7.3.20 *The following equalities hold:*

(i) $I_{\eta'_i} = \eta_d I_{\eta_{d-i}}$ *for all* i $(0 \le i \le d)$;

(ii) $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$;

(iii) $J_{[\alpha]} = J_{[\alpha^{-1}]}$;

(iv) $\tilde{J}^{(i)}_{[\alpha]} = \tilde{J}^{(d-i)}_{[\alpha^{-1}]}$ *for each* i $(0 \le i \le d)$.

Proof (i) Take $x \in \eta_d I_{\eta_{d-i}}$. Then $x = \eta_d y$ with $y \in I_{\eta_{d-i}}$ and $x\eta_{d-i} = \eta_{d-i} y \eta_d \in \eta_d R$. Hence $x(\eta_{d-i}/\eta_d) \in R$, which implies that $x \in I_{\eta_{d-i}/\eta_d} = I_{\eta'_i}$.

Conversely, take $x \in I_{\eta'_{d-i}}$. Hence $x\eta'_{d-i} = x(\eta_i/\eta_d) = (x/\eta_d)\eta_i \in R$ and so $x/\eta_d \in I_{\eta_i}$. Thus $x \in \eta_d I_{\eta_i}$.
(ii) $I_{[\alpha^{-1}]} = \bigcap_{j=0}^{d} I_{\eta'_j} = \bigcap_{j=0}^{d} \eta_d I_{\eta_j} = \eta_d I_{[\alpha]}$.
(iii) $J_{[\alpha]} = I_{[\alpha]}(\eta_0, \ldots, \eta_d) = \eta_d I_{[\alpha]}(\eta_0/\eta_d, \ldots, \eta_{d-1}/\eta_d, \eta_d/\eta_d) = I_{[\alpha^{-1}]}(\eta'_0, \ldots, \eta'_d) = J_{[\alpha^{-1}]}$.
(iv) is obtained by the same way as the proof of (ii). ∎

Lemma 7.3.21 (*cf.* Theorem 7.3.7 and its proof) *If α is exclusive over R, i.e., $R[\alpha] \cap K = R$, then either* $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ *or* $\tilde{J}_{[\alpha]} = R$.

Recall that if α is a super-primitive element (respectively an anti-integral element) over R then so is α^{-1} over R (*cf.* Theorems 2.4.5 and 2.4.8).

Proposition 7.3.22 *Assume that α is exclusive over R, i.e., $R[\alpha] \cap K = R$ and that R contains an infinite field. Then α (respectively α^{-1}) is exclusive over R, i.e., $R[\alpha] \cap K = R$ (respectively $R[\alpha^{-1}] \cap K = R$) if and only if either* $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ *(respectively* $\mathrm{grade}(\tilde{J}_{[\alpha^{-1}]}) > 1$*) or $\tilde{J}_{[\alpha]} = R$ (respectively $\tilde{J}_{[\alpha^{-1}]} = R$).*

Proof The implication ($\Rightarrow$) follows from Lemma 7.3.21. The reverse implication ($\Leftarrow$) follows from (Theorem 7.3.7). ∎

Proposition 7.3.23 *Assume that α is a super-primitive element over R. Then the following statements are equivalent for each i $(0 \le i \le d)$:*

(i) $\bigcap_{j \ne i} I_{\eta_j} \subseteq I_{\eta_i}$;
(i') $\bigcap_{j \ne d-i} I_{\eta'_j} \subseteq I_{\eta'_{d-i}}$;
(ii) $\mathrm{grade}(\tilde{J}^{(i)}_{[\alpha]}) > 1$ *or* $\tilde{J}^{(i)}_{[\alpha]} = R$;
(ii') $\mathrm{grade}(\tilde{J}^{(d-i)}_{[\alpha^{-1}]}) > 1$ *or* $\tilde{J}^{(d-i)}_{[\alpha^{-1}]} = R$.

Proof The equivalences: (i) $\Leftrightarrow$ (i′) and (ii) $\Leftrightarrow$ (ii′) are shown in Lemma 7.3.20.
(i) $\Rightarrow$ (ii): We assume that $\tilde{J}^{(i)}_{[\alpha]} \ne R$. Suppose that $\mathrm{grade}(\tilde{J}^{(i)}_{[\alpha]}) = 1$. Then there exists $p \in \mathrm{Dp}_1(R)$ such that $\tilde{J}^{(i)}_{[\alpha]} = I_{[\alpha]}(\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_d) \subseteq p$. Since $p \in \mathrm{Dp}_1(R)$, there exists an element $\beta \in K$ such that $I_\beta = p$. Thus $\beta J^{(i)}_{[\alpha]} = \beta I_{[\alpha]}(\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_d) \subseteq R$. Hence $(I_{[\alpha]}\beta)\eta_j \subseteq R$ for all $j \ne i$, which yields that $I_{[\alpha]}\beta \subseteq \bigcap_{j \ne i} I_{\eta_j}$. Since $\bigcap_{j \ne i} I_{\eta_j} \subseteq I_{\eta_i}$, we have that $I_{[\alpha]}\beta\eta_i \subseteq R$. Thus $J_{[\alpha]}\beta \subseteq R$, that is, $J_{[\alpha]} \subseteq I_\beta = P$. But since α is super-primitive over R, we have $J_{[\alpha]} \not\subseteq p$, which is a contradiction. So we conclude that $\mathrm{grade}(\tilde{J}^{(i)}_{[\alpha]} > 1$.

(ii) $\Rightarrow$ (i): First, consider the case $\tilde{J}^{(i)}_{[\alpha]} \neq R$. Suppose that there exists an element $a \in \bigcap_{j \neq i} I_{\eta_j}$ but $a \notin I_{\eta_i}$. Then $\beta := a\eta_i \notin R$. So $\beta I_{[\alpha]}(\eta_0, \eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_d) \subseteq R$. Since $\mathrm{grade}(\tilde{J}^{(i)}_{[\alpha]}) = \mathrm{grade}(I_{[\alpha]}(\eta_0, \eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_d)) > 1$, we have $\beta R_p = \beta(I_{[\alpha]}(\eta_0, \eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_d)R_p \subseteq R_p$ for every $p \in \mathrm{Dp}_1(R)$. Hence $\beta \in \bigcap_{p \in \mathrm{Dp}_1(R)} R_p = R$, which is a contradiction. Thus $\bigcap_{j \neq i} I_{\eta_j} \subseteq I_{\eta_i}$. Second, suppose that $\tilde{J}^{(i)}_{[\alpha]} = R$. Then there exists an equality: $1 = a_0\eta_0 + a_1\eta_1 + \cdots + a_{i-1} + a_{i+1}\eta_{i+1} + \cdots + a_d\eta_d$ with $a_i \in I_{[\alpha]}$. So $\eta_i = (a_0\eta_i)\eta_0 + (a_1\eta_i)\eta_1 + \cdots + (a_{i-1}\eta_i)\eta_{i-1} + (a_{i+1}\eta_i)\eta_{i+1} + \cdots + (a_d\eta_i)\eta_d$. Take $X \in \bigcap_{j \neq i} I_{\eta_j}$. Then $x\eta_i = x(a_0\eta_i)\eta_0 + x(a_1\eta_i)\eta_1 + \cdots + x(a_{i-1}\eta_i)\eta_{i-1} + x(a_{i+1}\eta_i)\eta_{i+1} + \cdots + x(a_d\eta_i)\eta_d \in R$, and hence $x \in I_{\eta_i}$. Therefore we conclude that $\bigcap_{j \neq i} I_{\eta_j} \subseteq I_{\eta_i}$.

The equivalence (i′) $\Leftrightarrow$ (ii′) can be shown by the same way as the proof of (i) $\Leftrightarrow$ (ii). ∎

Remark 7.3.24 Let A be an integral domain containing R. Then the following statements are equivalent:

(1) A is exclusive over R (i.e., $A \cap K = R$);

(2) A_p is exclusive over R_p (i.e., $A_p \cap K = R_p$) for all $p \in \mathrm{Dp}_1(R)$. [Indeed, we have only to recall that $\bigcap_{p \in \mathrm{Dp}_1(R)} R_p = R$.] ∎

Theorem 7.3.25 *Assume that α is a super-primitive element of degree d over R, that R contains an infinite field, and that* $\mathrm{grade}(I_{[\alpha]} + I_{[\alpha^{-1}]}) > 1$. *Then the following statements are equivalent:*

(i) $R[\alpha, \alpha^{-1}]$ *is exclusive over R, i.e.,* $R[\alpha, \alpha^{-1}] \cap K = R$;

(ii) $\mathrm{grade}(\tilde{J}^{(i)}_{[\alpha]}) > 1$ *or* $\tilde{J}^{(i)}_{[\alpha]} = R$ *for every* i $(0 \leq i \leq d)$;

(iii) $\bigcap_{j \neq i} I_{\eta_j} \subseteq I_{\eta_i}$ *for every* i $(0 \leq i \leq d)$;

(ii') $\mathrm{grade}(\tilde{J}^{(i)}_{[\alpha^{-1}]}) > 1$ *or* $\tilde{J}^{(i)}_{[\alpha^{-1}]} = R$ *for every* i $(0 \leq i \leq d)$;

(iii') $\bigcap_{j \neq i} I_{\eta'_j} \subseteq I_{\eta'_i}$ *for every* i $(0 \leq i \leq d)$.

Proof The equivalences (ii) $\Leftrightarrow$ (iii), (ii′) $\Leftrightarrow$ (iii′), and (ii) $\Leftrightarrow$ (ii′) follow from Proposition 7.3.23.

(i) $\Rightarrow$ (iii): Suppose that $\bigcap_{j \neq i} I_{\eta_j} \not\subseteq I_{\eta_i}$ for some i $(0 \leq i \leq d)$. Then there exists an element $a \in \bigcap_{j \neq i} I_{\eta_j}$ with $\lambda := a\eta_i \notin R$. Since $\varphi_\alpha(\alpha) = 0$, we have $a\varphi_\alpha(\alpha) = a\alpha^d + a\eta_1\alpha^{d-1} + \cdots + a\eta_i\alpha^{d-i} + \cdots + a\eta_d = 0$, which yields $-\lambda\alpha^{d-i} = a\eta_0\alpha^d + \cdots + a\eta_{i-1}\alpha^{d-i+1} + a\eta_{i+1}\alpha^{d-i-1} + \cdots + a\eta_d$. So we have $-\lambda = a\alpha^i + \cdots + (a\eta_{i-1})\alpha + (a\eta i + 1)\alpha^{-1} + \cdots + (a\eta_d)\alpha^{i-d} \in R[\alpha, \alpha^{-1}] \cap K = R$, which is a contradiction. Therefore we conclude that $\bigcap_{j \neq i} I_{\eta_j} \subseteq I_{\eta_i}$ for every i $(0 \leq i \leq d)$.

(ii') $\Rightarrow$ (i): Since $R[\alpha, \alpha^{-1}] \cap K = R \Leftrightarrow R_p[\alpha, \alpha^{-1}] \cap K = R_p$ for all $p \in \mathrm{Dp}_1(R)$ by Remark 7.3.24, we may assume that R is a local domain with maximal ideal p with $\mathrm{depth}(R) = 1$. In this case, $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$ because $\mathrm{grade}(I_{[\alpha]} + I_{[\alpha^{-1}]}) > 1$, and hence $I_{[\alpha]} = R$ or $I_{[\alpha^{-1}]} = R$. Take $\lambda \in R[\alpha, \alpha^{-1}] \cap K$. Then

$$(*) \quad \lambda = a_0\alpha^n + \cdots + a_{n-1}\alpha + a_n + b_1\alpha^{-1} + \cdots + b_m\alpha^{-m}$$

where $a_i, b_j \in R$. Since $\lambda \in R \Leftrightarrow \lambda - a_n \in R$, we may assume that $a_n = 0$. Put $\psi(X) := a_0X^{n+m} + \cdots + a_{n-1}X^{m+1} - \lambda X^m + b_1X^{m-1} + \cdots + b_m$. Then $\psi(\alpha) = 0$. Since α^{-1} is a super-primitive element by Lemma 7.3.20, we may assume that $I_{[\alpha]} = R$ by symmetry. Thus $\varphi_\alpha(X) \in R[X]$.
(a) Assume that $n = 0$. Then $\lambda \in R[\alpha^{-1}]$. Since $\tilde{J}_{[\alpha^{-1}]} = \tilde{J}^{(d)}_{[\alpha^{-1}]} = R$, it follows that α^{-1} is exclusive over R by Proposition 7.3.22. Hence $\lambda \in R[\alpha^{-1}] \cap K = R$.
(b) Assume that $m \geq d$. Then $\deg \psi(X) = n + m \geq d$. Since $\varphi_\alpha(X) \in R[X]$, there exists a polynomial $h(X) \in R[X]$ such that $\psi^*(X) := \psi(X) - \varphi_\alpha(X)h(X)$ is of degree m. Note that $\psi(X) \in R[X]$ if and only if $\psi^*(X) \in R[X]$. Hence using $\psi^*(X)$ instead of $\psi(X)$, we can assume that $n = 0$, which is done in (a).
(c) Assume that $m < d$. Then since $\varphi_\alpha(X) \in R[X]$, there exists a polynomial $h(X) \in R[X]$ such that $\psi^*(X) := \psi(X) - \varphi_\alpha(X)h(X)$ is of degree less than d. Note that $\psi^*(\alpha) = \psi(\alpha) - \varphi_\alpha(\alpha)h(\alpha) = 0$. It follows that $\psi^*(X) = 0$ in $R[X]$ because α is an element of degree d over K. Hence $\psi(X) = \varphi_\alpha(X)h(X) \in R[X]$, which means that $\lambda \in R$.

The implications: (i) $\Rightarrow$ (iii′) and (ii) $\Rightarrow$ (i) can be proved by the similar way to the above argument. ∎

Theorem 7.3.26 *Assume that* $\mathrm{grade}(I_{[\alpha]} + I_{[\alpha^{-1}]}) > 1$. *Then* $R[\alpha, \alpha^{-1}] \cap K = R$ *if and only if* $R[\alpha] \cap K = R$ *and* $R[\alpha^{-1}] \cap K = R$ *(i.e., both* α *and* α^{-1} *are exclusive over* R*).*

Proof ($\Leftarrow$): Take $p \in \mathrm{Dp}_1(R)$. Then $I_{[\alpha]} \not\subseteq p$ or $I_{[\alpha^{-1}]} \not\subseteq p$. by the assumption $\mathrm{grade}(I_{[\alpha]} + I_{[\alpha^{-1}]}) > 1$. Assume that $I_{[\alpha]} \not\subseteq p$. Then $(I_{[\alpha]})_p = R_p$, so that $\varphi_\alpha(X) \in R_p[X]$. Since $\varphi_\alpha(\alpha) = \alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0$, we have $\alpha = -(\eta_1 + \eta_2\alpha^{-1} + \cdots + \eta_d\alpha^{1-d}) \in R_p[\alpha^{-1}]$. Thus $R_p[\alpha] \subseteq R_p[\alpha^{-1}]$ and hence $R_p[\alpha, \alpha^{-1}] = R_p[\alpha^{-1}]$. So we have $R_p[\alpha, \alpha^{-1}] \cap K = R_p$ by the assumption α^{-1} is exclusive over R_p (*cf.* Remark 7.3.24). Next assume that $I_{[\alpha^{-1}]} \not\subseteq p$. Then in the similar way to the above case, we have $R_p[\alpha, \alpha^{-1}] \cap K = R_p$ by the assumption α is exclusive over R_p (*cf.* Remark 7.3.24). Thus we have $R \subseteq R[\alpha, \alpha^{-1}] \cap K \subseteq \bigcap_{p \in \mathrm{Dp}_1(R)} R_p[\alpha, \alpha^{-1}] \cap K = \bigcap_{p \in \mathrm{Dp}_1(R)} R_p = R$.

The implication ($\Rightarrow$) follows from the inclusions $R \subseteq R[\alpha^{\pm 1}] \cap K \subseteq R[\alpha, \alpha^{-1}] \cap K = R$. ∎

7.4 Finite Generation of an Intersection $R[\alpha] \cap K$ over R

Our objective is to investigate when the R-subalgebra $R[\alpha] \cap K$ is finitely generated over R.

In this section, we fix the following notation in addition to the definitions mentioned above unless otherwise specified:

Let R be a Noetherian domain with quotient field K. Let L be an algebraic field extension of K and α be an element in L which is of degree d over K. Let $\varphi_\alpha(X) := X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ denote the minimal polynomial of α over K (that is, $\eta_i \in K$). Let B be an intermediate ring between R and K. Put $I^B_{\eta_i} := B :_B \eta_i$, $I^B_{[\alpha]} := \bigcap_{i=1}^d I^B_{\eta_i}$, $J^B_{[\alpha]} := I^B_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$ and $\tilde{I}^B_{[\alpha]} := \bigcap_{i=1}^{d-1} I_{\eta_i^B}$. When $B = R$, $I^R_{[\alpha]} = I_{[\alpha]}$, $J^R_{[\alpha]} = J_{[\alpha]}$, and $\tilde{I}^R_{[\alpha]} = \tilde{I}_{[\alpha]}$. Note that if B is flat over R, then $I^B_{[\alpha]} = I_{[\alpha]}B$, $\tilde{I}^B_{[\alpha]} = \tilde{I}_{[\alpha]}B$, and $J^B_{[\alpha]} = J_{[\alpha]}B$.

We start with the following lemma:

Lemma 7.4.1 *Let t be an element in R.*

(1) $I_{[\alpha-t]} = I_{[\alpha]}$, $J_{[\alpha-t]} = J_{[\alpha]}$, *and* $\tilde{I}_{[\alpha-t]} = \tilde{I}_{[\alpha]}$.

(2) *Let $\eta_d^{(t)}$ denote the constant term of the minimal (monic) polynomial $\varphi_{\alpha-t}(X) \in K[X]$ of $\alpha - t$. Then $R[\tilde{I}_{[\alpha-t]}\eta_d^{(t)}] = R[\tilde{I}_{[\alpha]}\eta_d]$.*

Proof (1) By definition, we see that $I_{[\alpha]} \subseteq I_{[\alpha-t]}$ (respectively $J_{[\alpha]} \subseteq J_{[\alpha-t]}$, $\tilde{I}_{[\alpha]} \subseteq \tilde{I}_{[\alpha-t]}$). By symmetry, we also have $I_{[\alpha]} \subseteq I_{[\alpha-t]}$ (respectively $J_{[\alpha-t]} \subseteq J_{[\alpha]}$, $\tilde{I}_{[\alpha-t]} \subseteq \tilde{I}_{[\alpha]}$).
(2) It is easy to see that $\eta_d^{(t)} = \varphi_\alpha(t)$. We have $R[\tilde{I}_{[\alpha]}\eta_d] \supseteq R[\tilde{I}_{[\alpha]}\varphi_\alpha(t)] = R[\tilde{I}_{[\alpha]}\eta_d^{(t)}]$ by definition. By symmetry, we also have $R[\tilde{I}_{[\alpha]}\eta_d^{(t)}] \supseteq R[\tilde{I}_{[\alpha]}\eta_d]$. $\square$

An element α is called α *co-monic* over R if α^{-1} is integral over R. For a polynomial $f(X) \in R[X]$, we call $f(X)$ a co-monic polynomial over R if the constant term of $f(X)$ is 1.

The following theorem is the first main result of this section:

Theorem 7.4.2 *Assume that R contains an infinite field k, that $J_{[\alpha]} = R$, and that $\tilde{I}_{[\alpha]}$ is an invertible ideal. Then $R[\alpha] \cap K = R[\tilde{I}_{[\alpha]}\eta_d]$ and hence the R-algebra $R[\alpha] \cap K$ is finitely generated and flat over R.*

Proof Take $p \in \mathrm{Spec}(R)$. Since $J_{[\alpha]} = R$, $I_{[\alpha]}$ is invertible. Note that $\tilde{I}_{[\alpha]}$ is invertible by assumption. So it follows that $I_{[\alpha]}R_p = aR_p$ and $\tilde{I}_{[\alpha]} = bR_p$ for some $a, b \in R$. Note that $R_p = J_{[\alpha]}R_p = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)R_p =$

$a(1, \eta_1, \ldots, \eta_d)R_p$. Since k is an infinite field and since $J_{[\alpha]} = R \not\subseteq p$, there exists an element $t \in k$ such that $\varphi_\alpha(t) \neq 0$ and $a\varphi_\alpha(t) \in R \setminus p$. Let $\varphi_{\alpha-t}(X) := X^d + \eta_1^{(t)} + \cdots + \eta_d^{(t)}$. Since the constant term $\eta_d^{(t)}$ of the minimal polynomial $\varphi_{\alpha-t}(X)$ of $\alpha - t$ is given by $\varphi_\alpha(t)$, $\alpha - t$ is co-monic over R_p and in this case, $\eta_d^{(t)} = \varphi_\alpha(t)$ and $a\varphi_\alpha(t) \in R_p \setminus pR_p$. Since $\tilde{I}_{[\alpha-t]} = \tilde{I}_{[\alpha]} \supseteq I_{[\alpha]} = I_{[\alpha-t]}$ by Lemma 7.4.1, we have $bR_p \supseteq aR_p$ and hence $a = bc$ for some $c \in R_p$. So $\tilde{I}_{[\alpha-t]}\eta_d^{(p)} = bR_p\varphi_\alpha(t) = (a/c)\varphi_\alpha(t)R_p$. Hence $R_p[\tilde{I}_{[\alpha-t]}\eta_d^{(p)}] = R[(a/c)\varphi_\alpha(t)] = R_p[1/c] \subseteq R[\alpha]_p \cap K$ because $a\varphi_\alpha(t)$ is a unit in R_p. Since $R_p[1/c]$ is flat over R, we have $I_{[\alpha-t]}^{R_p[1/c]} = I_{[\alpha-t]}R_p[1/c]$, $\tilde{I}_{[\alpha-t]}^{R_p[1/c]} = \tilde{I}_{[\alpha-t]}R_p[1/c]$, and $J_{[\alpha]}^{R_p[1/c]} = J_{[\alpha]}R_p[1/c] = R_p[1/c]$. Hence α is super-primitive over $R_p[1/c]$. Since $\alpha - t$ is co-monic over $R_p[1/c]$, $\alpha - t$ is exclusive over $R_p[1/c]$, i.e., $R[\alpha]_p[1/c] \cap K = R_p[\alpha - t][1/c] \cap K = R_p[1/c]$ by Proposition 7.3.5. So we have $R[\alpha]_p[1/c] \cap K = R_p[1/c] \supseteq R[\alpha]_p \cap K \supseteq R_p[1/c]$. Therefore we obtain $R[\alpha]_p \cap K = R_p[\tilde{I}_{[\alpha-t]}\eta_d^{(p)}] = R_p[\tilde{I}_{[\alpha]}\eta_d]$ by Lemma 7.4.1. Since $p \in \mathrm{Spec}(R)$ is arbitrary, we conclude that $R[\alpha] \cap K = R[\tilde{I}_{[\alpha]}\eta_d]$. Moreover as was seen above, $R[\alpha]_p \cap K = R_p[\tilde{I}_{[\alpha]}\eta_d] = R_p[1/c]$ for each $p \in \mathrm{Spec}(R)$, which implies that $R[\alpha] \cap K$ is flat over R. ∎

Remark 7.4.3 In Corollary 7.4.13, we have shown that: Assume that α is a super-primitive element over R. If $P \cap R \not\supseteq I_{\eta_d} + \tilde{I}_{[\alpha]}$ for any $P \in \mathrm{Dp}_1(R[\tilde{I}_{[\alpha]}\eta_d]$, then $R[\alpha] \cap K = R[\tilde{I}_{[\alpha]}\eta_d]$. In particular, if $I_{\eta_d} + \tilde{I}_{[\alpha]} = R$, then $R[\alpha] \cap K = R[\tilde{I}_{[\alpha]}\eta_d]$. ∎

A related result is the following theorem, which is the second main result.

Theorem 7.4.4 *Assume that R contains an infinite field. Assume that α is a super-primitive element over R and that $R[\tilde{I}_{[\alpha]}\eta_d]$ is flat over R. Then $R[\alpha] \cap K = R[\tilde{I}_{[\alpha]}\eta_d]$ and hence $R[\alpha] \cap K$ is finitely generated over R.*

Proof Put $B := R[\tilde{I}_{[\alpha]}\eta_d]$. Since B is flat over R, we have $I_{\eta_i}^B = B :_B \eta_i = I_{\eta_i}B$. So $I_{[\alpha]}^B = I_{[\alpha]}B$, $\tilde{I}_{[\alpha]}^B = \tilde{I}_{[\alpha]}B$, and $J_{[\alpha]}^B = I_{[\alpha]}^B(1, \eta_1, \ldots, \eta_d) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)B = J_{[\alpha]}B$. Since α is super-primitive over R, $\mathrm{grade}(J_{[\alpha]}) > 1$. Since B is flat over R, $\mathrm{grade}(J_{[\alpha]}^B) > 1$, which means that α is super-primitive over B. Note that $\tilde{I}_{[\alpha]}^B\eta_d \subseteq B = R[\tilde{I}_{[\alpha]}\eta_d]$. So $\tilde{I}_{[\alpha]}^B \subseteq I_{\eta_d}^B$ and hence $\tilde{I}_{[\alpha]}^B \subseteq I_{[\alpha]}^B$. Thus α is exclusive over B by Theorem 7.4.7. Thus $B[\alpha] \cap K = B \supseteq R[\alpha] \cap K$. Take $b \in \tilde{I}_{[\alpha]}$. Then $b\eta_i \in R$ for all $1 \leq i \leq d-1$. So we have $b(\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_{d-1}\alpha) = -b\eta_d \in R[\alpha] \cap K$. Hence $B \subseteq R[\alpha] \cap K$. The reverse inclusion is obvious. ∎

Proposition 7.4.5 *Assume that R contains an infinite field. Assume that α is a super-primitive element over R and that there exists a Noetherian domain*

B such that $R[\tilde{I}_{[\alpha]}\eta_d] \subseteq B \subseteq R[\alpha] \cap K$, and that B is flat over R. Then $R[\alpha] \cap K = B$.

Proof Since B is flat over R, we have $I^B_{\eta_i} = B :_B \eta_i = I_{\eta_i} B$. So $I^B_{[\alpha]} = I_{[\alpha]}B$, $\tilde{I}^B_{[\alpha]} = \tilde{I}_{[\alpha]}B$, and $J^B_{[\alpha]} = I^B_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)B$. Note that $\tilde{I}^B_{[\alpha]}\eta_d \subseteq B = R[\tilde{I}_{[\alpha]}]$. So $\tilde{I}^B_{[\alpha]} \subseteq I^B_{\eta_d}$ and hence $\tilde{I}^B_{[\alpha]} \subseteq I^B_{[\alpha]}$. Thus α is exclusive over B by Theorem 7.3.7. Since α is a super-primitive element over R, we have $\mathrm{grade}(J_{[\alpha]}) > 1$. The flatness of B over R implies that $\mathrm{grade}(J^B_{[\alpha]}) = \mathrm{grade}(J_{[\alpha]}B)$. So α is super-primitive over B because so is $R[\alpha]$ over R. Hence by Theorem 7.3.7, we have $B[\alpha] \cap K = B \supseteq R[\alpha] \cap K \supseteq B$. Therefore $R[\alpha] \cap K = B$. ∎

The following proposition means that the associated prime divisors of D/R, where D is an integral domain containing R is determined by the birational extension $D \cap K$ of R.

Proposition 7.4.6 *Let D be a Noetherian integral domain containing R. Then* $\mathrm{Ass}_R(D \cap K/R) = \mathrm{Ass}_R(D/R)$.

Proof Put $B := D \cap K$. Since $R \subseteq B \subseteq D$, we have $\mathrm{Ass}_R(B/R) \subseteq \mathrm{Ass}_R(D/R)$. Conversely, take $p \in \mathrm{Ass}_R(D/R)$. Then there exists $\beta \in R[\alpha]$ such that $p = R :_R \beta$. Take a nonzero element $a \in p$. Then $a\beta \in R$ and hence $\beta \in K$, so that $\beta \in D \cap K = B$. Thus $p \in \mathrm{Ass}_R(B/R)$. ∎

Now recall the following lemma:

Lemma 7.4.7 (*cf.* Lemma 1.2.11). *Let C be an integral domain and D a subalgebra of the quotient field of C. Then $D \subseteq C$ is flat if and only if for every prime ideal p of D, either $pC = C$ or $D_p = C_p$.*

Proposition 7.4.8 *Let D be an integral domain containing R. Then $D \cap K$ is flat over R if and only if $p(D \cap K) = D \cap K$ for each $p \in \mathrm{Ass}_R(D \cap K/R)$.*

Proof Put $B := D \cap K$ and note that B is a birational extension of R.
($\Leftarrow$) Take $P \in \mathrm{Spec}(R)$. Assume that $R_P \neq B_P$. Then $R_P \not\supseteq B$. Choose an element $b \in B$ with $b \notin R_P$. Then $I_b := R :_R b \subseteq P$. Take a prime divisor p of I_b such that $p \subseteq P$. Then $p \in \mathrm{Ass}_R(B/R)$. By assumption, $pB = B$ and hence $PB = B$. By Lemma 7.4.7, B is flat over R.
($\Rightarrow$) Assume that B is flat over R. Then for $p \in \mathrm{Ass}_R(B/R)$, there exists $b \in B$ such that $p = I_b$. So we have $b \notin R_p$, which means that $B_p \neq R_p$. By Lemma 7.4.7, we conclude that $pB = B$. ∎

Remark 7.4.9 If there exists $p \in \mathrm{Ass}_R(R[\alpha]/R)$ such that $pR[\alpha] \neq R[\alpha]$, then $R[\alpha] \cap K$ is not flat over R.

The next theorem shows that flatness implies finiteness in the anti-integral extension, which is the third main result. ∎

Theorem 7.4.10 *Assume that α is an anti-integral element over R. If $R[\alpha] \cap K$ is flat over R, then $R[\alpha] \cap K$ is finitely generated over R.*

Proof Take a nonzero element $a \in I_{[\alpha]}$. Then α has a monic relation of degree d over $R[1/a]$. So $R[1/a][\alpha]$ is a free $R[1/a]$-module. Thus $B \subseteq R[1/a][\alpha] \cap K = R[1/a]$, which yields that $B/R \subseteq R[1/a]/R$ and that $\mathrm{Ass}_R(B/R) \subseteq \mathrm{Ass}(R[1/a]/R)$. So $\mathrm{Ass}_R(B/R)$ is a finite set because $\mathrm{Ass}_R(R[1/a]/R) \subseteq \{p \in \mathrm{Spec}(R) | p$ is a prime divisor of $aR\}$. Put $\mathrm{Ass}_R(B/R) = \{p_1, \ldots, p_\ell\}$ and put $I = p_1 \cap \cdots \cap p_\ell$. Since B is flat over R, we have $IB = B$ by Lemma 7.4.7 because $R_{p_i} \neq B_{p_i}$ for $1 \leq i \leq \ell$. So we can take elements $b_i \in I$ and $\beta_i \in B$ such that $1 = \sum_{i=1}^{\ell} b_i \beta_i$. Put $C = R[\beta_1, \ldots, \beta_\ell]$. The inclusion $C \subseteq B$ is obvious. Take $P \in \mathrm{Spec}(C)$ and put $P \cap R = p$. Then $I \not\subseteq p$ and $pB \neq B$. Since B is flat over R, we have $B_p = R_p$ by Lemma 7.4.7. Thus $B_p = C_p$. Since $B \subseteq C_P$, we have $B \subseteq \bigcap C_P = C$. Therefore $B = C$. ∎

Proposition 7.4.11 *Take $\beta \in K$. The following statements are equivalent:*

(i) *β is integral over $R[\alpha]$;*

(ii) *β is almost integral over $R[\alpha] \cap K$.*

Proof (i)⇒(ii): Since β is integral over $R[\alpha]$, β is almost integral over $R[\alpha]$. Put $I := \{x \in R[\alpha] | x\beta^\ell \in R[\alpha]$ for all $\ell > 0\} \subseteq R[\alpha]$, which is an ideal of $R[\alpha]$. Then $I \neq (0)$. Take $x \in I \setminus (0)$. Then there exists an algebraic relation: $a_0x^n + \cdots + a_{n-1}x + a_n = 0$ with $a_n \neq 0, a_i \in R[\alpha]$. In this case, $a_n \in I \cap R$, which means that $I \cap R \neq (0)$. Take $a \in I \cap R$ with $a_n \neq 0$. Then $a\beta^\ell \in R[\alpha] \cap K$. So β is almost integral over $R[\alpha] \cap K$.
(ii) ⇒ (i): Since β is almost integral over $R[\alpha] \cap K$, β is almost integral over $R[\alpha]$. Since $R[\alpha]$ is Noetherian, β is integral over $R[\alpha]$. ∎

An element $\beta \in K$ is called a *flat element* over R if $R[\beta]$ is flat over R.

Remark 7.4.12 In Corollary 7.4.17, we see: Assume that α is super-primitive over R and that $R[\eta_d]$ is flat over R. Then $R[\alpha] \cap K \subseteq R[\eta_d]$. ∎

Proposition 7.4.13 *Assume that R contains an infinite field and that α is a super-primitive element over R. If η_d is a flat element over R and η_d is integral over $R[\alpha]$, then $R[\alpha] \cap K = R[\eta_d]$.*

Proof The inclusion $R[\eta_d] \subseteq R[\alpha] \cap K$ follows from the above remark. Since $R[\eta_d]$ is flat over R, α is super-primitive over $R[\eta_d]$ by Proposition 7.1.15. By Theorem 7.3.7, we have $R[\eta_d][\alpha] \cap K = R[\eta_d]$. Thus $R[\alpha] \cap K \subseteq R[\alpha][\eta_d] \cap K = R[\eta_d][\alpha] \cap K = R[\eta_d]$. Therefore $R[\alpha] \cap K = R[\eta_d]$. $\square$

For an element $\beta \in K$, let J_β denote the ideal $J_{[\beta]}$, that is, $I_\beta(1, \beta)R$.

Proposition 7.4.14 *Assume that α is a super-primitive element over R. Assume that η_d is integral over $R[\alpha]$ and is super-primitive over R and that $J_{[\alpha]} + J_{\eta_d} = R$. Then $\eta_d \in R[\alpha] \cap K$.*

Proof Take $P \in \mathrm{Spec}(R[\alpha])$ and put $p = P \cap R$. Then by assumption, either $J_{[\alpha]} \not\subseteq p$ or $J_{\eta_d} \not\subseteq p$. Assume that $J_{[\alpha]} \not\subseteq p$.

Then $R[\alpha]_p$ is flat over R_p. So $J_{\eta_d} R[\alpha]_p \subseteq J_{\eta_d}^{R[\alpha]_p}$ and hence η_d is super-primitive over $R[\alpha]_p$. Since η_d is integral over $R[\alpha]_p$, we have $\eta_d \in R[\alpha]_p \subseteq R[\alpha]_P$. Assume now that $J_{\eta_d} \not\subseteq p$. Since $J_{\eta_d} = I_{\eta_d} + \eta_d I_{\eta_d}$, we have $I_{\eta_d} \not\subseteq p$ or $\eta_d I_{\eta_d} \not\subseteq p$. If $I_{\eta_d} \not\subseteq p$, then $\eta_d \in R_p \subseteq R[\alpha]_p \subseteq R[\alpha]_P$. If $\eta_d I_{\eta_d} \not\subseteq p$, then $\eta_d = 1/a$ for some $a \in pR_p$. Since $1/a$ is integral over $R[\alpha]_p$, by the same argument in the proof of Proposition 7.4.13, we have $\eta_d = 1/a \in R[\alpha]_p \subseteq R[\alpha]_P$. Hence $\eta_d \in R[\alpha]$. $\square$

Remark 7.4.15 If $\eta_d \eta_i \in R[\alpha]$ for all $1 \leq i \leq d-1$, then η_d is integral over $R[\alpha]$. Indeed, since $\alpha^d + \eta_1 \alpha^{d-1} + \cdots + \eta_d = 0$, we have $\eta_d^2 + (\alpha^d)\eta_d + (\eta_1 \eta_d \alpha^{d-1} + \cdots \eta_{d-1}\eta_d \alpha) = 0$. So η_d satisfies the monic relation of degree 2 over $R[\alpha]$. $\square$

Proposition 7.4.16 *Assume that R contains an infinite field, that α is an anti-integral element over R, and that η_d is integral over $R[\alpha]$ and anti-integral over R. If $J_{\eta_d} R[\alpha] = R[\alpha]$, then $\eta_d \in R[\alpha] \cap K$.*

Proof Take $P \in \mathrm{Spec}(R[\alpha])$ and put $p = P \cap R$. Then $J_{\eta_d} R[\alpha] = R[\alpha]$. So $J_{\eta_d} \not\subseteq p$. Since $J_{\eta_d} = I_{\eta_d} + \eta_d I_{\eta_d}$, we have either $J_{\eta_d} \not\subseteq p$ or $\eta_d I_{\eta_d} \not\subseteq p$. If $J_{\eta_d} \not\subseteq p$, then $\eta_d \in R_p \subseteq R[\alpha]_P$. If $\eta_d I_{\eta_d} \not\subseteq p$, then $\eta_d = 1/a$ for some $a \in pR_p$. Since η_d is integral over $R[\alpha]_p$, we have $\eta_d = 1/a \in R[\alpha]_p$ by the proof of Proposition 7.4.14. Hence $\eta_d \in R[\alpha]$. $\square$

7.5 Pure Extensions

Our main objective of this section is to show the following results:

(1) A pure extension of an integral domain is an exclusive extension.

(2) Let R be a ring and A be a ringextension of R. Then A is flat over R and A is a pure extension of R if and only if A/R is flat over R.

(3) If a simple extension $R[\alpha]$ of a Noetherian domain R is anti-integral and integral over R, then $R[\alpha]$ is a pure extension of R.

(4) Finally we give some examples.

We start with the following definition.

Definition 7.5.1 Let R be a ring and let $\phi : N \to M$ be an R-homomorphism of R-modules. We say that ϕ is pure if ϕ induces the injection $\phi \otimes_R 1_E : N \otimes_R E \to M \otimes_R E$ for every R-module E. When N is an R-submodule of M, we say that N is a *pure* R-submodule of M if the inclusion $i : N \hookrightarrow M$ induces the injection $i \otimes_R 1_E : N \otimes_R E \to M \otimes_R E$ for every R-module E.

Proposition 7.5.2 *Let R be a ring and let $\phi : N \to M$ be an R-homomorphism of R-modules. Then ϕ is pure over R if and only if $\phi \otimes_R 1_L : N \otimes_R L \to M \otimes_R L$ is injective for each finitely generated R-module L.*

Proof Any R-module E is obtained as an inductive limit of an inductive system: $E_1 \subseteq E_2 \subseteq \cdots$, where each E_i is a finitely generated R-module. Then we have $N \otimes_R E = N \otimes_R \varinjlim E_i = \varinjlim(N \otimes_R E_i)$ and $M \otimes_R E = M \otimes_R \varinjlim E_i = \varinjlim(M \otimes_R E_i)$, since $\phi \otimes_R 1_{E_i} : N \otimes_R E_i \to M \otimes_R E_i$ is injective for each i. Since the functor $\varinjlim$ is exact, it follows that $\varinjlim(N \otimes_R E_i) \to \varinjlim(M \otimes_R E_i)$ is injective. Hence $\phi \otimes_R 1_E : N \otimes_R E \to M \otimes_R E$ is injective, which means that ϕ is pure over R. $\square$

Proposition 7.5.3 *Let R be a ring and let $\phi : N \to M$ and $\psi : M \to L$ be R-homomorphisms of R-modules N, M, and L.*

(i) *If both ϕ and ψ are pure, then $\psi \cdot \phi$ is pure.*

(ii) *If $\psi \cdot \phi$ is pure, then ϕ is pure.*

(iii) *Let A be R-algebra. If ϕ is pure, then $\phi \otimes_R 1_A : N \otimes_R A \to M \otimes_R A$ is pure.*

Proof Let E be an R-module.
(i) Since $\phi \otimes_R 1_E$ and $\psi \otimes_R 1_E$ are injective, so is $(\psi \otimes_R 1_E)(\phi \otimes_R 1_E) = (\psi \cdot \phi) \otimes_R 1_E$. Hence $\psi \cdot \phi$ is pure.
(ii) Since $(\psi \cdot \phi) \otimes_R 1_E = (\psi \otimes_R 1_E)(\phi \otimes_R 1_E)$ is injective, $\phi \otimes_R 1_E : N \otimes_R E \to M \otimes_R E$ is injective. Hence ϕ is pure.
(iii) Let W be an A-module. Then W is regarded as an R-module naturally. Consider $(\phi \otimes_R 1_A) \otimes_A 1_W : (N \otimes_R A) \otimes_A W \to (M \otimes_R A) \otimes_A W$. Note that $(\phi \otimes_R 1_A) \otimes_A 1_W = \phi \otimes_R 1_W : N \otimes_R W \to M \otimes_R W$. Since ϕ is pure, $\phi \otimes_R 1_W$ is injective. Thus $\phi \otimes_R 1_A$ is pure. $\square$

Corollary 7.5.4 *Let R be a ring and let $\phi : N \to M$ be a pure R-homomorphism of R-modules N into M. Let I be an ideal of R. Then $\phi^{-1}(IM) = IN$.*

Proof Since ϕ is pure, $\phi \otimes_R 1_{R/I} : N/IN = N \otimes_R R/I \to M \otimes R/I = M/IM$ is injective. Hence $\phi^{-1}(IM) = IN$. $\square$

Remark 7.5.5 Let R be a ring and $N \subseteq M$ be R-modules.

(1) If $M = N \oplus L$ for some R-submodule L of M. Then N is a pure R-submodule of M.
(2) If M/N is flat over R, then N is a pure R-submodule of M. $\square$

Proposition 7.5.6 *Let R be a ring and let $\phi : N \to M$. Then the following statements are equivalent:*

(i) *ϕ is pure;*
(ii) *$\phi_p : N_p \to M_p$ is pure for every prime ideal p of R;*
(iii) *$\phi_m : N_m \to M_m$ is pure for every maximal ideal m of R.*

Proof (i) $\Rightarrow$ (ii): In Proposition 7.5.3, we have only to take $A = R_p$.
(ii) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$ (i): Let E be an R-module. Then for every maximal ideal m of R, $\mathrm{Ker}(\phi \otimes_R 1_E)_m = \mathrm{Ker}(\phi_m \otimes_{R_m} 1_{W_m}) = 0$ because ϕ_m is pure. Thus $\mathrm{Ker}(\phi \otimes_R 1_E) = 0$, which implies that ϕ is pure. $\square$

Definition 7.5.7 Let R be a ring and let A be an R-algebra. We say that A is a *pure extension* of R if R is a pure R-submodule of A.

The next proposition shows that a pure extension satisfies the Lying-Over Theorem.

Theorem 7.5.8 *Let R be a ring and let A be an R-algebra. If A is a pure extension of R, the canonical map* $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *is surjective.*

Proof Suppose that $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is not surjective. Then there exists a prime ideal p of R such that $pA \cap R \supsetneq p$. Since the canonical homomorphism $R_p \to A_p$ is pure by Proposition 7.5.6, we may replace R_p (resp. A_p) by R (resp. A). In this case, $pA = A$. Since A is a pure extension of R, we have the injection $R/p = R \otimes_R R/p \to A \otimes_R R/p = A/pA = 0$. Thus $R/p = 0$, a contradiction. Hence $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is surjective. ∎

Proposition 7.5.9 *Let R be a ring and let A be a pure extension of R. If A is a field, then R is a field.*

Proof Suppose that there exists a nonzero prime ideal p of R. Then the canonical homomorphism $R/p = R \otimes_R R/p \to A \otimes_R R/p = A/pA$ is injective because the inclusion $R \hookrightarrow A$ is pure. Thus $A/pA \neq 0$. Since A is a field and $p \neq (0)$, we have $pA = A$, a contradiction. Hence R is a field. ∎

Let A be a ring extension of an integral domain R and K denote the quotient field of R. We say that A is an *exclusive* extension of R if $A \cap K = R$. Let B be an intermediate ring between A and R. If A is an exclusive extension of R, then B is also an exclusive extension of R.

The following proposition shows that a pure extension is an exclusive extension.

Theorem 7.5.10 *Let R be an integral domain with quotient field K and let A be a pure extension of R. Then $A \cap K = R$. In particular, if $R \subseteq A \subseteq K$, then $A = R$.*

Proof Take $\gamma \in A \cap K$. We shall show that $\gamma \in R$. Take $a \in (R :_R \gamma) \setminus (0)$. Applying Corollary 7.5.4 to the case $I = aR$, $N = R$ and $M = A$, we have $aA \cap R = aR$. Since $a\gamma \in aA \cap R = aR$, it follows that $\gamma \in R$. The converse inclusion is obvious. ∎

Corollary 7.5.11 *Let R be an integral domain and let A be an exclusive extension of R. If an element $a \in R$ is a unit in A, then a is a unit in R.*

Proof If $a \in R$ is a unit in A, then there exists $\zeta \in A$ such that $a\zeta = 1$. Since $\zeta = 1/a \in K$, $\zeta \in A \cap K = R$. Hence a is a unit in R. $\square$

Corollary 7.5.12 *Let R be an integral domain and let A be a pure extension of R. If an element $a \in R$ is a unit in A, then a is a unit in R.*

Theorem 7.5.13 *Let R be a ring and let A be a ring extension of R. Then A is flat over R and A is a pure extension of R if and only if A/R is flat over R.*

Proof Let M be an R-module. We have the exact sequence:

$$(*) \qquad 0 \longrightarrow R \longrightarrow A \longrightarrow A/R \longrightarrow 0$$

Tensoring $- \otimes_R M$, we have the exact sequence:

$$(**) \quad \mathrm{Tor}_1^R(M, A) \to \mathrm{Tor}_1^R(M, A/R) \to R \otimes_R M \xrightarrow{\psi} A \otimes_R M \to \cdots$$

($\Rightarrow$): Since A is flat over R, we have $\mathrm{Tor}_1^R(M, A) = 0$. Since A is a pure extension of R, ψ is injective. Hence $\mathrm{Tor}_1^R(M, A/R) = 0$ by $(**)$, which means that A/R is flat over R (*cf.* [M2]).
($\Leftarrow$): Since R and A/R are flat over R, the exact sequence $(*)$ yields that A is flat over R (*cf.* [M2]). Next the exact sequence $(**)$ yields that the canonical homomorphism $R \otimes_R M \to A \otimes_R M$ is injective because $\mathrm{Tor}_1^R(M, A/R) = 0$. Thus A is a pure extension of R. $\square$

In the rest of this section, we consider the following situation. Let R be a Noetherian domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic field extension L of the quotient field K of R and $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$$

Then η_i $(1 \le i \le d)$ are uniquely determined by α. Let $I_{\eta_i} := R :_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$, the latter of which is called a *denominator ideal* of α. We say that α is an anti-integral element if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. For $f(X) \in R[X]$, let $\mathsf{c}(f(X))$ denote the ideal of R generated by the coefficients of $f(X)$. For an ideal J of $R[X]$, let $\mathsf{c}(J)$ denote the ideal generated by the coefficients of the elements in J. If α is an anti-integral element, then $\mathsf{c}(\mathrm{Ker}(\pi)) = \mathsf{c}(I_{[\alpha]}\varphi_\alpha(X)R[X]) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Put $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) | \mathrm{depth}(R_p) = 1\}$, then α is called a super-primitive element. It is known that a super-primitive element is an anti-integral

element (*cf.* Chapter 2). By definition, the super-primitive is characterized by the set of $\mathrm{Dp}_1(R)$.

Proposition 7.5.14 *If α is anti-integral and integral over R, then $R[\alpha]$ is a pure extension of R.*

Proof The extension $R[\alpha]$ is a flat R-module by Theorem 2.3.2, and hence $R[\alpha] = R \oplus \alpha R \oplus \cdots \oplus \alpha^{d-1} R$ by Theorem 2.3.8. Thus the inclusion $R \hookrightarrow R[\alpha]$ is pure by Remark 7.5.5. ∎

Theorem 7.5.15 *Assume that R is a Noetherian normal domain and that α is integral over R. Then $R[\alpha]$ is a flat, pure extension of R.*

Proof Since R is a Krull domain, α is a super-primitive element over R by Theorem 2.2.9. So our conclusion follows from Proposition 7.5.14. ∎

Example 7.5.16 Let k be an infinite field and t be an indeterminate. Put $R := k[t]$, a polynomial ring.

(i) Let α be an element satisfying the following algebraic relation:

$$\alpha^2 + \frac{1}{t}\alpha + \frac{1}{t^2} = 0$$

Since R is a Noetherian normal domain, α is a super-primitive element by Theorem 2.2.9. It is easy to see that $I_{[\alpha]} = t^2 R$ and $J_{[\alpha]} = I_{[\alpha]}(1, 1/t, 1/t^2)R = R$. So $R[\alpha]$ is a flat extension of R by Theorem 2.3.6 but that α is not integral over R by Theorem 2.3.2. By Theorem 7.3.7, $R[\alpha]$ is not an exclusive extension of R, and hence $R[\alpha]$ is not a pure extension of R by Theorem 7.5.10.

(ii) Let α be an element satisfying the following algebraic relation:

$$\alpha^2 + \frac{1}{t}\alpha + \frac{1}{t} = 0 \quad \cdots (*)$$

Since R is a Noetherian normal domain, α is a super-primitive element. It is easy to see that $I_{[\alpha]} = tR$ and $J_{[\alpha]} = I_{[\alpha]}(1, 1/t, 1/t)R = R$. So $R[\alpha]$ is a flat extension of R but that α is *not* integral over R. By Lemma 7.3.7, $R[\alpha]$ is an exclusive extension of R. Consider the R-module $R[\alpha]/R$. Since $t\alpha^2 + \alpha + 1 = 0$ in $R[\alpha]$, we have $\overline{\alpha} = -t\overline{\alpha^2}$ in $R[\alpha]/R$. Thus $\overline{\alpha^i} R \subseteq \overline{\alpha^{i+1}} R$ for all $i \geq 1$. Let $H_n := \overline{\alpha} R + \cdots + \overline{\alpha^{n-1}} R \subseteq R[\alpha]/R$.

Then $H_n \subseteq \overline{\alpha^n}R$. Suppose that there exist $a, b \in R$ ($a \neq 0, b \neq 0$) such that $a(b\overline{\alpha^n}) = \overline{0}$. Then $ab\alpha^n \in R$ and hence $\alpha^n \in K$. So by $(*)$, we have $\alpha^{n-1} \in K$. Using $(*)$ successively, we have $\alpha \in K$, which contradicts $d = 2$. So H_n is a torsion-free R-submodule of $R[\alpha]/R$. Since R is a PID $k[t]$, H_n is a free R-submodule of $R[\alpha]/R$. Hence $R[\alpha]/R = \varinjlim H_n$ is a flat R-module. Hence $R[\alpha]$ is a pure extension of R by Theorem 7.5.13.

Chapter 8

Ultra-Primitive Extensions and Their Generators

Let R be a Noetherian domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic extension field L of the quotient field K of R and $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$$

Then η_i $(1 \leq i \leq d)$ are uniquely determined by α. Let $I_{\eta_i} := R :_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$. If $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)$, we say that α is *anti-integral* over R. Put $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Then $J_{[\alpha]} = \mathsf{c}(I_{[\alpha]}\varphi_\alpha(X))$, where $\mathsf{c}(\ )$ denotes the ideal generated by the coefficients of the polynomials in $(\ )$, that is, the content ideal of $(\ )$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) \mid \mathrm{depth}(R_p) = 1\}$, then α is called a *super-primitive element* over R. It is known that a super-primitive element over R is anti-integral over R (*cf.* Theorem 2.2.8). It is also known that any algebraic element over a Krull domain R is anti-integral over R (*cf.* Theorem 2.2.9). When α is an element in K, $\varphi_\alpha(X) = X - \alpha$. So we have $J_{[\alpha]} = I_{[\alpha]}(1, \alpha) = I_\alpha(1, \alpha) = I_\alpha + \alpha I_\alpha = I_\alpha + I_{\alpha^{-1}}$, where $I_\alpha := R :_R \alpha$. In this case, $\alpha \in K$ is anti-integral over R if and only if $R = R\langle\alpha\rangle =: R\langle\alpha\rangle$.

Throughout this chapter, we use the following notation unless otherwise specified: Let R be a Noetherian domain with quotient field K and $\overline{R}$ denote the integral closure of R in K. Then $\overline{R}$ is a Krull domain (*cf.* [M2). Let $\mathcal{C}(\overline{R}/R)$ denote the conductor ideal, i.e., $\{a \in R | a\overline{R} \subseteq R\}$. Let L be an algebraic extension field of K and α be an element in L which is of degree d over K. Let $\varphi_\alpha(X) := X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ denote the minimal polynomial of α over K (that is, $\eta_i \in K$). Let I be an ideal of R. When $I \neq R$, then we define $\mathrm{grade}(I) :=$ the length of a maximal R-sequence in I. When $I = R$, let $\mathrm{grade}(I) := \infty$ by convention (*cf.* [M2, p.106]). Note in advance that $K[\alpha]$ is an algebraic extension field and $K[\alpha] = K(\alpha)$.

8.1 Super-Primitive Elements and Ultra-Primitive Elements

We say that R is "N-1" if $\overline{R}$ is a finite R-module (*cf.* [M1, (32.A)]). Let $\mathcal{C}(\overline{R}/R)$ denote the conductor ideal, i.e., $\{a \in R | a\overline{R} \subseteq R\}$. If R is N-1, then $\mathcal{C}(\overline{R}/R) \neq 0$. We say that R is "N-2" if, for any finite extension L' of K, the integral closure $R_{L'}$ of R in L' is a finite module. Let B be an intermediate ring between R and L. Assume that B is integral over R. If R is N-2, then B is N-1. If R is N-1 (resp. N-2), so is any localization of R (*cf.* [M1, (32.A)]). It is known that when R is of characteristic zero, R is N-1 if and only if R is N-2 (*cf.* [M1, (32.B)]).

As shown in the examples below, a sum (a product) of anti-integral elements is not necessarily anti-integral. So we introduce some stronger property and we want to find a set of elements with such a property that is closed in addition and multiplication. This is our objective of this section.

An element $\alpha \in L$ is called to be an *ultra-primitive element* over R if α satisfies

$$\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$$

If R is not N-1, then $\mathcal{C}(\overline{R}/R)$ may be (0). In this case, if $\alpha \in K$ is ultra-primitive over R, then the condition $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$ implies that $I_{[\alpha]} = R$ because $\mathrm{grade}(I_{[\alpha]}) = 1$ or $I_{[\alpha]} = R$, and hence that $\alpha \in R$.

Theorem 8.1.1 implies that an ultra-primitive element is a super-primitive element and hence an anti-integral element.

Theorem 8.1.1 *If $\alpha \in L$ is ultra-primitive over R, then it is super-primitive over R.*

Proof In order to prove that α is super-primitive over R, it suffices that α is super-primitive over R_p for every $p \in \mathrm{Dp}_1(R)$ by Theorem 2.2.8. Take $p \in \mathrm{Dp}_1(R)$. Then $I_{[\alpha]} \not\subseteq p$ or $\mathcal{C}(\overline{R}/R) \not\subseteq p$ by assumption. When $I_{[\alpha]} \not\subseteq p$, α has a monic algebraic relation of degree d over R_p. So α is anti-integral over R_p by Propositions 2.1.2 and 2.1.3. Since $I_{[\alpha]} \subseteq J_{[\alpha]}$, we have $J_{[\alpha]} \not\subseteq p$. Thus α is super-primitive over R. When $\mathcal{C}(\overline{R}/R) \not\subseteq p$, $\overline{R}_p = R_p$ holds. Hence α is super-primitive over R_p by Theorem 2.2.9. $\square$

Remark 8.1.2 Assume that R is normal (i.e., integrally closed in K). Then any algebraic element over R is ultra-primitive over R by definition. $\square$

Example 8.1.3 Assume that R is N-1, that is, $\overline{R}$ is a finitely generated R-module. So $\mathcal{C}(\overline{R}/R) \neq (0)$.

(1) Take $a \in \mathcal{C}(\overline{R}/R) \setminus (0)$. Then $\alpha := 1/a$ is anti-integral but not ultra-primitive over R. In fact $R[\alpha] \cap R[\alpha^{-1}] = R \cap R[\alpha^{-1}] = R$. Since $I_{[\alpha]} + \mathcal{C}(\overline{R}/R) = I_\alpha + \mathcal{C}(\overline{R}/R) = \mathcal{C}(\overline{R}/R)$, $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) = 1$.

(2) Take $\gamma \in \overline{R} \setminus R$. Then γ is not anti-integral over R. Indeed, suppose that γ is anti-integral over R. Then $R[\gamma]$ is a free R-module of rank 1 since γ is integral over R. Thus $R[\gamma] = R$, a contradiction. Put $\gamma = a/b$ with $a, b \in R$. Then $\alpha := 1/b$, and $\beta := a$. Then α and β are anti-integral over R. In fact, $R[\alpha] \cap R[\alpha^{-1}] = R[1/a] \cap R[a] = R[1/a] \cap R = R$ and $a \in R$. But $\alpha\beta = \gamma$ is not anti-integral over R.

Example 8.1.4 Let k be a field and $k[t]$ be a polynomial ring. Let $R := k[t^2, t^3]_{(t^2,t^3)}$. Then $\alpha := (t^3 + 1)/t^2$ and $\beta := -1/t^2$ are anti-integral over R. In fact, $R[\alpha] \cap R[\alpha^{-1}] = R[1/t^2] \cap R[t^2] = R$ and $R[\beta] \cap R[\beta^{-1}] = R[1/t^2] \cap R[t^2] = R[1/t^2] \cap R = R$. But $\alpha + \beta = t$ is not anti-integral over R.

Remark 8.1.5 If α is anti-integral and integral over R, then $R[\alpha]$ is a free R-module $R \oplus R\alpha \oplus \cdots \oplus R\alpha^{d-1}$ (*cf.* Theorem 2.3.8). $\square$

Let $R \subseteq S$ be Noetherian domains, let $\overline{S}$ be the integral closure of S in its quotient field $K(S)$ and $R[X]$ and $S[X]$ be polynomial rings. Let $f(X), g(X) \in K(S)[X]$ and put $I_f^{(S)} := S[X] :_S f(X)$, $I_g^{(S)} := S[X] :_S g(X)$, $I_{fg}^{(S)} := S[X] :_S f(X)g(X)$, which are ideals of S, and $I_{fg}^{(R)} := R[X] :_R f(X)g(X)$, which is an ideal of R.

Lemma 8.1.6 *We employ the above notation. Assume that S is N-1 and that $f(X), g(X) \in K[X]$ ($K = K(R)$). Then the following statements hold:*

(1) $I_{fg}^{(R)} \subseteq I_{fg}^{(S)} \cap R$;

(2) $\sqrt{I_f^{(S)} I_g^{(S)}} = \sqrt{I_{fg}^{(S)}} = \sqrt{I_f^{(S)}} \cap \sqrt{I_g^{(S)}}$;

(3) $\sqrt{\mathsf{c}(I_f^{(S)} I_g^{(S)} f(X)g(X))} \supseteq \sqrt{\mathsf{c}(I_{fg}^{(R)} f(X)g(X))}$.

Proof (1) $I_{fg}^{(R)} = R[X] :_R f(X)g(X) \subseteq S[X] :_R f(X)g(X) = (S[X] :_S f(X)g(X)) \cap R = I_{fg}^{(S)} \cap R$.
(2) Let $\overline{S}$ denote the integral closure of S in $K(S)$. Then $\overline{S}$ is a Noetherian normal domain and $\mathcal{C}(\overline{S}/S) \neq (0)$ since S is N-1.

Step 1: We first show: $\sqrt{I_f^{(\overline{S})} I_g^{(\overline{S})}} = \sqrt{I_{fg}^{(\overline{S})}}$.

Proof. The inclusion $I_f^{(\overline{S})} I_g^{(\overline{S})} \subseteq I_{fg}^{(\overline{S})}$ is obvious. Take $P \in \mathrm{Ht}_1(\overline{S})$. Then $\overline{S}_P$ is a DVR and hence a unique factorization domain (UFD). So $(I_f^{(\overline{S})} I_g^{(\overline{S})})_P = (I_{fg}^{(\overline{S})})_P$ by Gauss' Lemma. Since $I_{fg}^{(\overline{S})}$ is divisorial, we have $\sqrt{I_f^{(\overline{S})} I_g^{(\overline{S})}} \supseteq \sqrt{I_{fg}^{(\overline{S})}}$. Thus $\sqrt{I_f^{(\overline{S})} I_g^{(\overline{S})}} = \sqrt{I_{fg}^{(\overline{S})}}$. It is obvious that $\sqrt{I_f^{(\overline{S})}} \cap \sqrt{I_g^{(\overline{S})}} = \sqrt{I_f^{(\overline{S})} I_g^{(\overline{S})}}$.

Step 2: We shall show that $\sqrt{I_f^{(\overline{S})}} \cap S = \sqrt{I_f^{(S)}}$, $\sqrt{I_g^{(\overline{S})}} \cap S = \sqrt{I_g^{(S)}}$, and $\sqrt{I_{fg}^{(\overline{S})}} \cap S = \sqrt{I_{fg}^{(S)}}$.

Proof. First note that $I_f^{(S)} = S[X] :_S f(X)$ and $I_f^{(\overline{S})} = \overline{S}[X] :_{\overline{S}} f(X)$. Hence $I_f^{(\overline{S})} \cap S = \overline{S}[X] :_S f(X) \supseteq S[X] :_S f(X) = I_f^{(S)}$. Conversely, take $P \in \mathrm{Dp}_1(S)$. Assume that $\mathcal{C}(\overline{S}/S) \not\subseteq P$. Then $\overline{S}_P = S_P$. Since $(I_f^{(\overline{S})})_P = I_f^{(\overline{S}_P)}$ and $(I_f^{(S)})_P = I_f^{(S_P)}$, we have $\sqrt{(I_f^{(\overline{S})})_P} \cap S_P = \sqrt{(I_f^{(S)})_P}$. Next assume that $\mathcal{C}(\overline{S}/S) \subseteq P$. Note that $\sqrt{\mathcal{C}(\overline{S}/S)} S_P = P S_P$ by the choice of P.

Take $c \in \sqrt{I_f^{(\overline{S})}} \cap S$. Then $c \in P S_P = \sqrt{\mathcal{C}(\overline{S}/S)} S_P$ and hence $c^\ell \in \mathcal{C}(\overline{S}/S)_P \cap (I_f^{(\overline{S})})_P$. Thus $c^\ell f(X) \in S_P[X]$. So we have $c^\ell \in (I_f^{(S)})_P$, which shows that $(\sqrt{I_f^{(\overline{S})}} \cap S)_P \subseteq \sqrt{(I_f^{(S)})_P}$. Therefore we have $\sqrt{I_f^{(\overline{S})}} \cap S \subseteq \sqrt{I_f^{(S)}}$ because $I_f^{(S)}$ is divisorial. So $\sqrt{I_f^{(\overline{S})}} \cap S = \sqrt{I_f^{(S)}}$. By the same argument, we have $\sqrt{I_g^{(\overline{S})}} \cap S = \sqrt{I_g^{(S)}}$ and $\sqrt{I_{fg}^{(\overline{S})}} \cap S = \sqrt{I_{fg}^{(S)}}$.

Step 3: We have

$$\begin{aligned}
\sqrt{I_f^{(S)} I_g^{(S)}} &= \sqrt{\left(\sqrt{I_f^{(\overline{S})}} \cap S\right)\left(\sqrt{I_g^{(\overline{S})}} \cap S\right)} \\
&= \sqrt{\left(\sqrt{I_f^{(\overline{S})}} \cap S\right) \cap \left(\sqrt{I_g^{(\overline{S})}} \cap S\right)} = \sqrt{\sqrt{I_f^{(\overline{S})}} \cap \sqrt{I_g^{(\overline{S})}} \cap S} \\
&= \sqrt{\sqrt{I_f^{(\overline{S})}} \sqrt{I_g^{(\overline{S})}} \cap S} = \sqrt{\sqrt{I_f^{(\overline{S})} I_g^{(\overline{S})}} \cap S} \\
&= \sqrt{I_f^{(\overline{S})} I_g^{(\overline{S})}} \cap S = \sqrt{I_{fg}^{(\overline{S})}} \cap S = \sqrt{I_{fg}^{(S)}}
\end{aligned}$$

by Step 1 and Step 2.

(3) follows from (1) and (2). $\square$

Lemma 8.1.7 *Let S be a Noetherian N-1 domain with $R \subseteq S \subseteq L$ and let $S[X]$ be a polynomial ring. Let $f(X), g(X) \in K(S)[X]$ be nonzero polynomials such that $f(X)g(X) \in K[X]$. If $\mathsf{c}(I_{fg}^{(R)} f(X)g(X)) = R$, then $\mathsf{c}(I_f^{(S)} f(X)) = \mathsf{c}(I_g^{(S)} g(X)) = S$.*

Proof Since $\mathsf{c}(I_{fg}^{(S)} f(X)g(X)) \cap R \supseteq \mathsf{c}(I_{fg}^{(R)} f(X)g(X)) = R$ by Lemma 8.1.6(1), we have $\mathsf{c}(I_{fg}^{(S)} f(X)g(X)) = S$ and hence $\mathsf{c}(I_f^{(S)} I_g^{(S)} f(X)g(X)) = S$ by Lemma 8.1.6(3). Note that $\mathsf{c}(f(X))^\ell \mathsf{c}(f(X)g(X)) = \mathsf{c}(f(X))^{\ell+1}\mathsf{c}(g(X))$ for a large ℓ (*cf.* [G, (28.1)]). Thus

$$\begin{aligned}
&\mathsf{c}(I_f^{(S)} f(X))^\ell \mathsf{c}(I_f^{(S)} I_g^{(S)} f(X)g(X)) \\
&= (I_f^{(S)})^{\ell+1}(I_g^{(S)})\mathsf{c}(f(X))^\ell \mathsf{c}(f(X)g(X)) \\
&= (I_f^{(S)})^{\ell+1}(I_g^{(S)})\mathsf{c}(f(X))^{\ell+1}\mathsf{c}(g(X)) \\
&= \mathsf{c}(I_f^{(S)} f(X))^{\ell+1}\mathsf{c}(I_g^{(S)} g(X))
\end{aligned}$$

So we have $\mathsf{c}(I_f^{(S)} f(X))^\ell = \mathsf{c}(I_f^{(S)} f(X))^{\ell+1}\mathsf{c}(I_g^{(S)} g(X))$. Suppose that $\mathsf{c}(I_f^{(S)} f(X))\mathsf{c}(I_g^{(S)} g(X)) \neq S$. Then

$$\mathsf{c}(I_f^{(S)} f(X))^\ell = (\mathsf{c}(I_f^{(S)} f(X))\mathsf{c}(I_g^{(S)} g(X)))\mathsf{c}(I_f^{(S)} f(X))^\ell$$

yields that there exists $c \in \mathsf{c}(I_f^{(S)} f(X))\mathsf{c}(I_g^{(S)} g(X))$ such that $(1 + c)\mathsf{c}(I_f^{(S)} f(X))^\ell = (0)$. Since S is an integral domain, we have $\mathsf{c}(I_f^{(S)} f(X))^\ell = (0)$, a contradiction. Hence $\mathsf{c}(I_f^{(S)} f(X)) = \mathsf{c}(I_g^{(S)} g(X)) = S$. ∎

Let B be a Noetherian domain such that $R \subseteq B \subseteq L$ and $K(B)$ denote the quotient field of B. Let $\varphi_\alpha^{K(B)}(X) := X^\ell + \eta_1^{(B)} X^{\ell-1} + \cdots + \eta_\ell^{(B)}$ be the monic minimal polynomial of $\alpha \in L$ over $K(B)$. Put $I_{[\alpha]}^B := \bigcap_{i=1}^\ell I_{\eta_i^{(B)}}^B$ $(= B[X] :_B \varphi_\alpha^{K(B)}(X))$, where $I_{\eta_i^{(B)}}^B := B :_B \eta_i^{(B)}$. Put $J_{[\alpha]}^B := I_{[\alpha]}^B(1, \eta_1^{(B)}, \ldots, \eta_\ell^{(B)})(= \mathsf{c}(I_\alpha^B \varphi_\alpha^{K(B)}(X)))$.

Let B be an integral domain containing R. Define

$$\mathrm{Dp}_1(B) \cap R := \{P \cap R | P \in \mathrm{Dp}_1(B)\}$$

Proposition 8.1.8 *Let B be a Noetherian domain such that $R \subseteq B \subseteq L$ and that $\mathrm{Dp}_1(B) \cap R \subseteq \mathrm{Dp}_1(R)$. Assume that R is N-2. If α is super-primitive over R, then α is super-primitive over B.*

Proof Take $P \in \mathrm{Dp}_1(B)$. Then $p := P \cap R \in \mathrm{Dp}_1(R)$ by the assumption. Localizing at p, we may assume that $J_{[\alpha]} = R$ because α is super-primitive

over R. So we have only to prove that $J^B_{[\alpha]} = B$. Since $\varphi_\alpha^{K(B)}(X)|\varphi_\alpha(X)$ in $K(B)[X]$. Put $\varphi_\alpha(X) = \varphi_\alpha^{K(B)}(X)\phi(X)$ in $K(B)[X]$. In this case, $\mathsf{c}(I^B_{fg}f(X)g(X)) = \mathsf{c}(B[X] :_B \varphi_\alpha(X)) \supseteq \mathsf{c}(R[X] :_R \varphi_\alpha(X))B = J_{[\alpha]}B = B$ and $\phi_0(X) \in K(B)[X]$ is monic. Then we can apply Lemma 8.1.7 for $S := B$, $f(X) := \phi(X) \in K(B)[X]$, and $g(X) := \varphi_\alpha^{K(B)}(X) \in K(B)[X]$. Thus we have $J^B_{[\alpha]} = \mathsf{c}(I^B_{[\alpha]}\varphi_\alpha^{K(B)}(X))$ and $\mathsf{c}((I^B_{[\alpha]}\varphi_\alpha^{K(B)}(X))) = \mathsf{c}(I^B_g g(X)) = B$ by Lemma 8.1.7 because $\mathsf{c}(I^{(R)}_{fg}f(X)g(X)) = \mathsf{c}(I^{(R)}_{\varphi_\alpha}\varphi_\alpha(X)) = J_{[\alpha]} = R$, and so $J^B_{[\alpha]} = \mathsf{c}(I^B_{[\alpha]}\varphi_\alpha^{K(B)}(X)) = B$. Therefore we conclude that α is super-primitive over B. ∎

Lemma 8.1.9 *Let I, J, and T be ideals of R such that* $\mathrm{grade}(I + T) > 1$ *and* $\mathrm{grade}(J + T) > 1$. *Then* $\mathrm{grade}(IJ + T) > 1$.

Proof Let a_1, a_2 (respectively b_1, b_2) be a regular sequence in $\sqrt{I} + T$ (respectively $\sqrt{J} + T$). Note that $(\sqrt{I} + T)(\sqrt{J} + T) \subseteq \sqrt{IJ} + T$. We have only to show that at least two elements in $\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$ form a regular sequence in $\sqrt{IJ} + T$, but this is straightforward. ∎

Lemma 8.1.10 *Let α, β be elements in L. Assume that R is N-2.*

(1) *Assume that* $\mathrm{Dp}_1(R[\alpha + \beta]) \cap R \subseteq \mathrm{Dp}_1(R)$. *Then* $\sqrt{I_{[\alpha]}I_{[\beta]}} \subseteq \sqrt{I_{[\alpha+\beta]}}$.

(2) *Assume that* $\mathrm{Dp}_1(R[\alpha\beta]) \cap R \subseteq \mathrm{Dp}_1(R)$. *Then* $\sqrt{I_{[\alpha]}I_{[\beta]}} \subseteq \sqrt{I_{[\alpha\beta]}}$.

Proof Take $p \in \mathrm{Dp}_1(R)$. Suppose that $I_{[\alpha]}I_{[\beta]} \not\subseteq p$. Then $I_{[\alpha]} \not\subseteq p$ and $I_{[\beta]} \not\subseteq p$. Since $I_{[\alpha]} \subseteq J_{[\alpha]}$ (respectively $I_{[\beta]} \subseteq J_{[\beta]}$), α (respectively β) is super-primitive over R_p. So α and β are both integral over R_p by Theorem 2.3.2, and hence $\alpha + \beta$ and $\alpha\beta$ are integral over R_p. Since R is N-2, both $R_p[\alpha + \beta]$ and $R_p[\alpha\beta]$ are N-1. Thus α is super-primitive and integral over $R_p[\alpha + \beta]$ and $R_p[\alpha\beta]$ by Proposition 8.1.8. Put $\gamma := \alpha + \beta$. Hence $I^{R_p[\gamma]}_{[\alpha]} = R_p[\gamma]$ and so $I^{R_p[\gamma]}_{[\alpha]} \cap R_p = R_p[\gamma] \cap R_p = R_p$. Let $\varphi_\gamma^{K[\alpha]}(X) \in K[\alpha][X]$ denote the minimal monic polynomial of γ over $K[\alpha]$. We have that $R_p[\alpha][\beta] = R_p[\gamma][\alpha]$ is a free $R_p[\gamma]$-module $R_p[\gamma] \oplus R_p[\gamma]\alpha \oplus \cdots \oplus R_p[\gamma]\alpha^{\ell-1}$, where $\ell = \deg \varphi_\alpha^{K[\gamma]}(X)$. In particular,

$$(*) \quad K[\gamma][\alpha] = K[\gamma] \oplus K[\gamma]\alpha \oplus \cdots \oplus K[\gamma]\alpha^{\ell-1}$$

Put $\varphi_\gamma^{K[\alpha]}(X) = \phi_0(X) + \phi_1(X)\alpha + \cdots + \phi_{d-1}(X)\alpha^{\ell-1}$ with $\phi_i(X) \in K[X]$, where $\phi_0(X)$ is monic in $K[X]$ with $\deg \phi_0(X) \leq \deg \varphi_\gamma^{K[\alpha]}(X) \leq \deg \varphi_\gamma(X)$. Note that $\phi_0(\gamma) + \phi_1(\gamma)\alpha + \cdots + \phi_{d-1}(\gamma)\alpha^{d-1} = \varphi_\gamma^{K[\alpha]}(\gamma) = 0$, which is a direct sum in $(*)$. Hence $\phi_0(\gamma) = 0$. So we have $\varphi_\gamma(X)|\phi_0(X)$. Hence $\varphi_\gamma(X) =$

$\phi_0(X)$. Thus $R_p = I_{[\beta]}^{R_p[\alpha]} \cap R_p = I_{[\gamma]}^{R_p[\alpha]} \cap R_p = R_p[\alpha] :_{R_p} \varphi_\gamma^{K[\alpha]}(X) \subseteq R_p :_{R_p} \phi_0(X) = R_p :_{R_p} \varphi_\gamma(X) = I_{[\gamma]}^{R_p}$, that is, $I_{[\alpha+\beta]}^{R_p} = I_{[\gamma]}^{R_p} = R_p$. Similarly $R_p = I_{[\beta]}^{R_p[\alpha]} \cap R_p \subseteq I_{[\beta\alpha]}^{R_p[\alpha]} \cap R_p = R_p[\alpha] :_{R_p} \varphi_{\beta\alpha}^{K[\alpha]}(X) \subseteq R_p :_{R_p} \varphi_{\beta\alpha}(X) = I_{[\beta\alpha]}^{R_p}$, that is, $I_{[\beta\alpha]}^{R_p} = R_p$. So we have $I_{[\alpha+\beta]} \nsubseteq p$ and $I_{[\alpha\beta]} \nsubseteq p$, which means that $\sqrt{I_{[\alpha]}I_{[\beta]}} \subseteq \sqrt{I_{[\alpha+\beta]}}$ and that $\sqrt{I_{[\alpha]}I_{[\beta]}} \subseteq \sqrt{I_{[\alpha\beta]}}$ because $I_{[\alpha]}$, $I_{[\beta]}$, $I_{[\alpha+\beta]}$, and $I_{[\alpha\beta]}$ are divisorial ideals in R by definition. ∎

Corollary 8.1.11 *Let α, β be elements in L and let $a \in R \setminus (0)$. Assume that R is N-2. Assume that both α and β are ultra-primitive over R.*

(1) *Assume that* $\mathrm{Dp}_1(R[\alpha + \beta]) \cap R \subseteq \mathrm{Dp}_1(R)$. *Then* $\mathrm{grade}(I_{[\alpha+\beta]} + \mathcal{C}(\overline{R}/R)) > 1$, *i.e., $\alpha + \beta$ is ultra-primitive over R.*

(2) *Assume that* $\mathrm{Dp}_1(R[\alpha\beta]) \cap R \subseteq \mathrm{Dp}_1(R)$. *Then* $\mathrm{grade}(I_{[\alpha\beta]} + \mathcal{C}(\overline{R}/R)) > 1$, *i.e., $\alpha\beta$ is ultra-primitive over R.*

(3) $\mathrm{grade}(I_{[a\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$, *i.e., $a\alpha$ is ultra-primitive over R.*

Proof By Lemma 8.1.10, we have $\sqrt{I_{[\alpha]}I_{[\beta]}} \subseteq \sqrt{I_{[\alpha+\beta]}}$ (resp. $\sqrt{I_{[\alpha]}I_{[\beta]}} \subseteq \sqrt{I_{[\alpha\beta]}}$). Hence $\mathrm{grade}(I_{[\alpha+\beta]} + \mathcal{C}(\overline{R}/R)) = \mathrm{grade}(\sqrt{I_{[\alpha+\beta]}} + \mathcal{C}(\overline{R}/R)) \geq \mathrm{grade}(\sqrt{I_{[\alpha]}I_{[\beta]}} + \mathcal{C}(\overline{R}/R)) > 1$ (resp. $\mathrm{grade}(I_{[\alpha\beta]} + \mathcal{C}(\overline{R}/R)) = \mathrm{grade}(\sqrt{I_{[\alpha\beta]}} + \mathcal{C}(\overline{R}/R)) \geq \mathrm{grade}(\sqrt{I_{[\alpha]}I_{[\beta]}} + \mathcal{C}(\overline{R}/R)) > 1$) by Lemma 8.1.9. (3) follows from that $I_{[a]}I_{[\alpha]} = I_{[\alpha]}$ because $a \in R$ (and hence $I_{[\alpha]} = I_a = R$). ∎

Lemma 8.1.12 *Let $R \hookrightarrow B \hookrightarrow A$ be Noetherian domains. Assume that A is integral and flat over R. Then B is flat over R.*

Proof Let $0 \to N \to M$ be an exact sequence of finitely generated R-modules, Let $\phi : N \otimes_R B \to M \otimes_R B$ be the canonical B-homomorphism induced from $N \to M$. Note that $(N \otimes_R B) \otimes_B A = N \otimes_R A$ and $(M \otimes_R B) \otimes_B A = M \otimes_R A$. Since A is flat over R, the canonical homomorphism $N \otimes_R A \to M \otimes_R A$ is injective. Hence $\phi \otimes_B A$ is injective. Put $T := \mathrm{Ker}(\phi)$ and let $\psi : T \hookrightarrow N \otimes_R B$. Suppose that $T \neq 0$. Since $\phi \otimes_B A$ is injective, we have $\psi \otimes_B A = 0$ and $\mathrm{Ann}_B(\psi) \neq B$. So it follows that $\mathrm{Ann}_B(\psi)A = A$. But since A is integral over B, we have $\mathrm{Ann}_B(\psi)A \neq A$, a contradiction. Hence $T = 0$ and hence $\phi : N \otimes_R B \to M \otimes_R B$ is injective, which means that B is flat over R. ∎

Theorem 8.1.13 *Let B be a Noetherian domain such that $R \subseteq B \subseteq L$. Assume that R is N-2 and that B is flat and integral over R. Let a be an element*

in R and let α and β be elements in B, which are ultra-primitive over R. Then so are $a\alpha$, $\alpha+\beta$, and $\alpha\beta$.

Proof We have only to show that $\mathrm{Dp}_1(R[\alpha+\beta])\cap R \subseteq \mathrm{Dp}_1(R)$. Take a prime ideal p of R with $\mathrm{grade}(p) \geq 2$. Then p contains a regular sequence a_1, a_2 of R. Since $R[\alpha+\beta]$ is flat over R by Lemma 8.1.12, a sequence a_1, a_2 is a regular sequence of $R[\alpha+\beta]$. Hence every prime ideal P of $R[\alpha+\beta]$ lying over p is of grade ≥ 2. Hence we conclude that $\mathrm{Dp}_1(R[\alpha+\beta])\cap R \subseteq \mathrm{Dp}_1(R)$. Similarly, we can show that $\mathrm{Dp}_1(R[\alpha\beta]) \cap R \subseteq \mathrm{Dp}_1(R)$. Hence $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$ and $\mathrm{grade}(I_{[\beta]} + \mathcal{C}(\overline{R}/R)) > 1$ imply that $\mathrm{grade}(I_{[\alpha+\beta]} + \mathcal{C}(\overline{R}/R)) > 1$ and $\mathrm{grade}(I_{[\alpha\beta]} + \mathcal{C}(\overline{R}/R)) > 1$ by Corollary 8.1.11. ∎

Since $I_{[a]} = I_a = R$ for any $a \in R$, every element in R is ultra-primitive over R by the definition of grade. Note here that $\mathrm{grade}(R) = \infty$ by definition. Hence we have the following results from Theorem 8.1.13.

Corollary 8.1.14 *Let B be a Noetherian domain such that $R \subseteq B \subseteq L$. Assume that R is N-2 and that B is flat and integral over R. The set D of all elements in B which are ultra-primitive over R forms an intermediate ring between R and B.*

Corollary 8.1.15 *Assume that R is N-2 and that α is an ultra-primitive element and integral over R. Then every element in $R[\alpha]$ is ultra-primitive over R.*

Proof The ring $R[\alpha]$ is a free R-module by Remark 8.1.5 because an ultra-primitive element is also an anti-integral element. Our conclusion follows Theorem 8.1.13. ∎

Remark 8.1.16 If $\mathcal{C}(\overline{R}/R) = (0)$, then an element in L which is ultra-primitive over R is in $\overline{R}$. In fact, if $\alpha \in L$ is ultra-primitive over R, then the condition $\mathrm{grade}(I_{[\alpha]}+\mathcal{C}(\overline{R}/R)) > 1$ implies that $\mathrm{grade}(I_{[\alpha]}) > 1$. But this yields $I_{[\alpha]} = R$ because $I_{[\alpha]}$ is divisorial. Hence $\alpha \in \overline{R}$. ∎

Proposition 8.1.17 *Let B be a Noetherian domain such that $R \subseteq B \subseteq L$. Assume that R is N-2 and that B is flat and integral over R. Let $\alpha_1, \ldots, \alpha_n \in B$ be ultra-primitive over R. Then every element in $R[\alpha_1, \ldots, \alpha_n]$ is ultra-primitive over R.*

Proof This follows Theorem 8.1.13. ∎

8.2 Comparisons of Subrings of Type $R[a\alpha] \cap R[(a\alpha)^{-1}]$

In this section, we use the following notations unless otherwise specified: Let R be a Noetherian domain with quotient field K, $\overline{R}$ denote the integral closure of R in K, and $\mathcal{C}(\overline{R}/R)$ denote the conductor, i.e., $\{a \in R | a\overline{R} \subseteq R\}$. Let L be an algebraic field extension of K and α be an element in L which is of degree d over K. Let $\varphi_\alpha(X) := X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ denote the minimal polynomial of α over K. Put $R\langle\alpha\rangle := R[\alpha] \cap R[\alpha^{-1}]$.

Let $a \in R$ be a nonzero element. Let $\varphi_{a\alpha}(X) := X^d + \eta_1^{(a)} X^{d-1} + \cdots + \eta_d^{(a)}$, the minimal monic polynomial of $a\alpha$ over K. Then $\varphi_{a\alpha}(X) = X^d + a\eta_1 X^{d-1} + \cdots + a^d \eta_d$. Hence $\eta_i^{(a)} = a^i \eta_i$ $(1 \le i \le d)$. So we have $I_{\eta_i} \subseteq I_{a^i \eta_i} = I_{\eta_i^{(a)}}$ and hence $I_{[\alpha]} \subseteq I_{[a\alpha]}$.

Lemma 8.2.1 *Let a be a nonzero element of R such that a is not a zero-divisor on $R/I_{[\alpha]}$. Then $I_{[a\alpha]} = I_{[\alpha]}$.*

Proof By definition, we have

$$I_{[a\alpha]} = \bigcap_{i=1}^{d} I_{a^i \eta_i} = \bigcap_{i=1}^{d} (I_{\eta_i} :_R a^i) \subseteq \bigcap_{i=1}^{d} (I_{\eta_i} :_R a^d) = \left(\bigcap_{i=1}^{d} I_{\eta_i} \right) :_R a^d = I_{[\alpha]} : a^d$$

Thus $I_{[\alpha]} :_R a^d = I_{[\alpha]}$ and hence $I_{[a\alpha]} \subseteq I_{[\alpha]}$ because a is not a zero-divisor in $R/I_{[\alpha]}$ (and hence so is a^d). Since $I_{[a\alpha]} \supseteq I_{[\alpha]}$ is shown above, we have $I_{[a\alpha]} = I_{[\alpha]}$. ∎

Proposition 8.2.2 *Let a be a nonzero element of R which is not a zero-divisor on $R/I_{[\alpha]}$. If α is super-primitive over R, then $a\alpha$ is also super-primitive over R.*

Proof Take $p \in \mathrm{Dp}_1(R)$. Assume that $p \in \mathrm{Ass}_R(R/I_{[\alpha]})$. Since $I_{[\alpha]}$ is a divisorial ideal of R, we have $\mathrm{depth}(R_p) = 1$. Since α is a super-primitive element over R, $J_{[\alpha]} \not\subseteq p$. Put $\varphi_{a\alpha}(X) = X^d + \eta_1^{(a)} X^{d-1} + \cdots + \eta_d^{(a)}$, which is the minimal polynomial of $a\alpha$. Then $\eta_i^{(a)} = a^i \eta_i$ $(1 \le i \le d)$. Suppose that $a^i(I_{[\alpha]}\eta_i) \subseteq p$ for all i. Note here that $I_{[\alpha]}\eta_i \subseteq R$. Then we have $(I_{[\alpha]})\eta_i \subseteq p$ for all i since $a \notin p$, which means that $J_{[\alpha]} \subseteq p$, a contradiction. So there exists i such that $a^i(I_{[\alpha]}\eta_i) \not\subseteq p$. Hence by using Lemma 8.2.1,

$$J_{[a\alpha]} = I_{[a\alpha]}\left(1, \eta_1^{(a)}, \ldots, \eta_d^{(a)}\right) = I_{[\alpha]}\left(1, a\eta_1, \ldots, a^d \eta_d\right) \not\subseteq p$$

Next assume that $p \notin \mathrm{Ass}_R(R/I_{[\alpha]})$. Then $R_p = (I_{[\alpha]})_p \subseteq (I_{[a\alpha]})_p \subseteq (J_{[a\alpha]})_p$, which means that $J_{[a\alpha]} \not\subseteq p$. Therefore we conclude that $a\alpha$ is super-primitive over R. ∎

Recall that an element $\alpha \in L$ is called ultra-primitive over R if $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$.

Proposition 8.2.3 *Assume that α is ultra-primitive over R. Then α is anti-integral over R.*

Proof In order to prove that α is anti-integral over R, it suffices that α is anti-integral over R_p for every $p \in \mathrm{Dp}_1(R)$ by Theorem 2.4.3. Take $p \in \mathrm{Dp}_1(R)$. Then $I_{[\alpha]} \not\subseteq p$ or $\mathcal{C}(\overline{R}/R) \not\subseteq p$ by assumption. When $I_{[\alpha]} \not\subseteq p$, α has a monic algebraic relation of degree d over R_p. So α is anti-integral over R_p by Theorem 2.3.8. When $\mathcal{C}(\overline{R}/R) \not\subseteq p$, We have $\overline{R}_p = R_p$. Hence α is anti-integral over R_p by Theorem 2.2.9. ∎

Corollary 8.2.4 *Assume that α is ultra-primitive over R. Then for any nonzero element $a \in R$, $a\alpha$ is anti-integral over R.*

Proof It is clear from the paragraph preceding Lemma 8.2.1 that $I_{[\alpha]} \subseteq I_{[a\alpha]}$. Then we see that α implies that $\mathrm{grade}(I_{[a\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$. Thus $a\alpha$ is anti-integral over R by Proposition 8.2.3. ∎

Remark 8.2.5 Assume that $d = 1$. If α and $a\alpha$ are both anti-integral over R, we have $R\langle\alpha\rangle = R[a\alpha] \cap R[(a\alpha)^{-1}] = R$. ∎

Proposition 8.2.6 *Let a be a nonzero element in R. Assume that $d \geq 2$ and that α and $a\alpha$ are anti-integral over R. Then $R[a\alpha] \cap R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle$ if and only if $aI_{[a\alpha]} \subseteq I_{[\alpha]}$.*

Proof Put $\zeta_i = \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i$ for i $(1 \leq i \leq d-1)$. Then it follows that $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ and that $R[a\alpha] \cap R[(a\alpha)^{-1}] = R \oplus I_{[a\alpha]}a\zeta_1 \oplus \cdots \oplus I_{[a\alpha]}a^{d-1}\zeta_{d-1}$ by Theorem 3.1.19 because both α and $a\alpha$ are anti-integral over R. Thus $R[a\alpha] \cap R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle$ if and only if $I_{[a\alpha]}a^i\zeta_i \subseteq I_{[\alpha]}\zeta_i$ for all i. This means that it is equivalent to $I_{[a\alpha]}a^i \subseteq I_{[\alpha]}$ for all i $(1 \leq i \leq d-1)$. Hence $aI_{[a\alpha]} \subseteq I_{[\alpha]}$. ∎

Corollary 8.2.7 *Let a be a nonzero element in R. Assume that $d \geq 2$ and that α is ultra-primitive over R. Then $R[a\alpha] \cap R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle$ if and only if $aI_{[a\alpha]} \subseteq I_{[\alpha]}$.*

Proof This follows Corollary 8.2.4 and Propositions 8.2.3 and 8.2.6. ∎

Proposition 8.2.8 *Let a be a nonzero element in R. Assume that both α and $a\alpha$ are anti-integral over R. Assume moreover that a is not a zero-divisor on R/I_{η_i} for any i $(1 \leq i \leq d)$. Then $R[a\alpha] \cap R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle$.*

Proof Since a is not a zero-divisor on R/I_{η_i} by the assumption, we have $I_{a^i\eta_i} = I_{\eta_i} :_R a^i = I_{\eta_i}$, and hence $I_{[a\alpha]} \subseteq I_{[\alpha]}$. The reverse inclusion is obvious. Hence $I_{[a\alpha]} = I_{[\alpha]}$. In particular, $aI_{[a\alpha]} = aI_{[\alpha]} \subseteq I_{[\alpha]}$. ∎

Corollary 8.2.9 *Let a be a nonzero element in R. Assume that α is ultra-primitive over R. Further assume that a is not a zero-divisor on R/I_{η_i} for any i $(1 \leq i \leq d)$. Then $R[a\alpha] \cap R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle$.*

Proof This follows from Corollary 8.2.4 and Proposition 8.2.8. ∎

Remark 8.2.10 Let a be a nonzero element in R. Assume that both α and $a\alpha$ are anti-integral over R. Then $I_{[a\alpha]} = \bigcap_{i=1}^{d} I_{a^i\eta_i} = \bigcap_{i=1}^{d}(I_{\eta_i} :_R a^i) \subseteq \bigcap_{i=1}^{d}(I_{\eta_i} :_R a^d) = (\bigcap_{i=1}^{d} I_{\eta_i}) :_R a^d = I_{[\alpha]} :_R a^d$. Hence $a^d I_{[a\alpha]} \subseteq I_{[\alpha]}$. In particular, $a^{d-1} I_{[a\alpha]} \subseteq I_{[\alpha]} :_R a$. If a is not a zero-divisor on $R/(I_{[\alpha]} :_R a)$, then $I_{[a\alpha]} \subseteq I_{[\alpha]} :_R a$. ∎

By the argument in the above remark, we extend Proposition 8.2.8 to the following theorem.

Proposition 8.2.11 *Let a be a nonzero element in R. Assume that both α and $a\alpha$ are anti-integral over R. Further assume that a is not a zero-divisor on $R/(I_{[\alpha]} : Ra)$. Then $R[a\alpha] \cap R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle$.*

Proof Since a is not a zero-divisor on $R/(I_{[\alpha]} :_R a)$, we have $aI_{[a\alpha]} \subseteq I_{[\alpha]}$ by the remark above. So $R[a\alpha] \cap R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle$ by Proposition 8.2.8. ∎

Corollary 8.2.12 *Let a be a nonzero element in R. Assume that α is ultra-primitive over R. Assume moreover that a is not a zero-divisor on $R/(I_{[\alpha]} :_R a)$. Then $R[a\alpha] \cap R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle$.*

Proof This follows from Corollary 8.2.4 and Proposition 8.2.11. ∎

Lemma 8.2.13 *Let I be a nonzero ideal of R and let a be an element of R. Then $aI = I$ if and only if a is a unit in R.*

Proof Suppose that a is a nonunit of R, and satisfies $aI = I$. Let $p \in \mathrm{Spec}(R)$ such that $a \in p$. Then $aI_p = I_p$, and so $I_p = (0)$ by Nakayama's (NAK) Lemma (*cf.* [M1]). Since R is an integral domain, we have $I = (0)$, a contradiction. ∎

Theorem 8.2.14 *Let a be a nonzero element in R. Assume that $d \geq 2$ and that both α and $a\alpha$ are anti-integral over R. Assume moreover that α is integral over R. Then the following statements are equivalent:*

(1) $R[a\alpha] \cap R[(a\alpha)^{-1}] = R\langle\alpha\rangle$;

(2) $R[a\alpha] = R[\alpha]$;

(3) $a^i I_{[a\alpha]} = I_{[\alpha]}$ *for all* i $(1 \leq i \leq d-1)$;

(4) *a is a unit in R.*

Proof Since α is integral over R, we have $I_{[\alpha]} = R$ and $I_{[a\alpha]} = R$. By Theorem 2.3.8, $R[\alpha]$ and $R[a\alpha]$ are free R-modules, that is,

$$R[\alpha] = R \oplus R\alpha \oplus \cdots \oplus R\alpha^{d-1}$$

and

$$R[a\alpha] = R \oplus Ra\alpha \oplus \cdots \oplus Ra^{d-1}\alpha^{d-1}$$

(2) $\Rightarrow$ (4): By the above direct sum decompositions, we have $Ra^i\alpha^i = R\alpha^i$ for all i $(1 \leq i \leq d-1)$. Hence $aR = R$.
(4) $\Rightarrow$ (2) is obvious.

Since α and $a\alpha$ are anti-integral over R, we have:

$$R[a\alpha] \cap R[(a\alpha)^{-1}] = R \oplus aI_{[a\alpha]}\zeta_1 \oplus \cdots \oplus a^{d-1}I_{[a\alpha]}\zeta_{d-1}$$

and

$$R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$$

where ζ_i's are the same as in the proof of Proposition 8.2.6.
(3)$\Leftrightarrow$ (1): $a^i I_{[a\alpha]}\zeta_i = I_{[\alpha]}\zeta_i$ for all i $(1 \leq i \leq d-1)$ if and only if $R[a\alpha] \cap R[(a\alpha)^{-1}] = R\langle\alpha\rangle$.
(4) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (4): Note that $a^i R \supseteq a^i I_{[a\alpha]} = I_{[\alpha]}$ for all i $(1 \leq i \leq d-1)$. Since α is integral over R, we have $I_{[\alpha]} = R$. Hence $aR = R$, which yields that a is a unit in R. ∎

Corollary 8.2.15 *Let a be a nonzero element in R. Assume that $d \geq 2$ and that α is ultra-primitive over R. Assume further that α is integral over R. Then the following statements are equivalent:*

(1) $R[a\alpha] \cap R[(a\alpha)^{-1}] = R\langle\alpha\rangle$;
(2) $R[a\alpha] = R[\alpha]$;
(3) $a^i I_{[a\alpha]} = I_{[\alpha]}$ *for all* i $(1 \leq i \leq d-1)$;
(4) *a is a unit in R.*

Proof This follows from Corollary 8.2.4 and Theorem 8.2.14. ∎

Theorem 8.2.16 *Let a be a nonzero element in R. Assume that both α and $a\alpha$ are anti-integral over R. If $d \geq 3$, then the following statements are equivalent:*

(1) $R[a\alpha] \cap R[(a\alpha)^{-1}] = R\langle\alpha\rangle$;
(2) $R[a\alpha] = R[\alpha]$ *and* $R[(a\alpha)^{-1}] = R[\alpha^{-1}]$;
(3) $a^i I_{[a\alpha]} = I_{[\alpha]}$ *for all* i $(1 \leq i \leq d-1)$;
(4) *a is a unit in R.*

Proof (1) $\Leftrightarrow$ (3): Since α and $a\alpha$ are anti-integral over R, we have:

$$R[a\alpha] \cap R[(a\alpha)^{-1}] = R \oplus aI_{[a\alpha]}\zeta_1 \oplus \cdots \oplus a^{d-1} I_{[a\alpha]}\zeta_{d-1}$$

and

$$R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$$

where ζ_i's are the same as defined in the proof of Proposition 8.2.6. Hence $a^i I_{[a\alpha]}\zeta_i = I_{[\alpha]}\zeta_i$ for all i $(1 \leq i \leq d-1)$. Thus $a^i I_{[a\alpha]} = I_{[\alpha]}$ for all i $(1 \leq i \leq d-1)$.
The reverse argument yields (3) $\Rightarrow$ (1).
(4) $\Rightarrow$ (2) is obvious.
(4) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (4): Since $d \geq 3$, $I_{[\alpha]} = aI_{[a\alpha]} = a^2 I_{[a\alpha]}$ by assumption, we have $I_{[a\alpha]} = aI_{[a\alpha]}$. Hence a is a unit in R by Lemma 8.2.13.
(2) $\Rightarrow$ (1) is trivial. ∎

Theorem 8.2.17 *Let a be a nonzero element in R. Assume that both α and $a\alpha$ are* ultra-primitive *over R and α is also ultra-primitive over $R\langle\alpha\rangle$. If $d \geq 3$,*

then the following statements are equivalent:

(i) $R\langle a\alpha\rangle = R\langle\alpha\rangle$ *i.e.,* $R[a\alpha] \cap R[(a\alpha)^{-1}] = R[\alpha] \cap R[\alpha^{-1}]$*;*

(ii) $R[\alpha] = R[a\alpha]$*;*

(iii) $a^i I_{[a\alpha]} = I_{[\alpha]}$ *for all* i $(1 \le i \le d-1)$*;*

(iv) a *is a unit in* R*.*

Proof Since α and $a\alpha$ are ultra-primitive over R, they are super-primitive over R by Theorem 8.1.1 and hence they are also anti-integral over R.
(i) $\Leftrightarrow$ (iii): $R[a\alpha] \cap R[(a\alpha)^{-1}] = R[\alpha] \cap R[\alpha^{-1}]$ if and only if $a^i I_{[a\alpha]} = I_{[\alpha]}$ for all i $(1 \le i \le d-1)$.
(iv) $\Rightarrow$ (ii) is obvious.
(iv) $\Rightarrow$ (iii) is trivial.
(iii) $\Rightarrow$ (iv): Since $d \ge 3$ and $I_{[\alpha]} = aI_{[a\alpha]} = a^2 I_{[a\alpha]}$, we have $I_{[a\alpha]} = aI_{[a\alpha]}$. Hence a is a unit in R.
(ii) $\Rightarrow$ (iv): Note that $a\alpha$ is ultra-primitive over R by Proposition 8.1.3. So α and $a\alpha$ are anti-integral over $R\langle\alpha\rangle$. Since $R\langle\alpha\rangle$ and $R[\alpha]$ are birational, we have $R\langle\alpha\rangle[a\alpha] \cap R\langle\alpha\rangle[(a\alpha)^{-1}] = R\langle\alpha\rangle$. It is clear that the equality $R\langle\alpha\rangle[\alpha] \cap R\langle\alpha\rangle[\alpha^{-1}] = R\langle\alpha\rangle$ always holds without the assumption that α is anti-integral over R. Since $a \in R$, we have $I_{[\alpha]} \subseteq I_{[a\alpha]}$. Thus $a\alpha$ is ultra-primitive over $R\langle\alpha\rangle$ because α is ultra-primitive over $R\langle\alpha\rangle$. So $R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle[(a\alpha)^{-1}] \subseteq R[\alpha^{-1}][(a\alpha)^{-1}] = R[(a\alpha)^{-1}]$, and hence $R\langle\alpha\rangle[(a\alpha)^{-1}] = R[(a\alpha)^{-1}]$. Since $R[\alpha] \subseteq R\langle\alpha\rangle[\alpha] \subseteq R[\alpha]$, we have $R\langle\alpha\rangle[\alpha] = R[\alpha]$. Note that $R\langle\alpha\rangle[\alpha] = R\langle\alpha\rangle[a\alpha]$ because $R[\alpha] = R[a\alpha]$ by the assumption. Thus we have: $R[a\alpha] \cap R[(a\alpha)^{-1}] = R[\alpha] \cap R[(a\alpha)^{-1}] = R\langle\alpha\rangle[\alpha] \cap R\langle\alpha\rangle[(a\alpha)^{-1}] = R\langle\alpha\rangle[a\alpha] \cap R\langle\alpha\rangle[(a\alpha)^{-1}] = R\langle\alpha\rangle$. Note that the minimal monic polynomial of $a\alpha$ is $\varphi_{a\alpha}(X) = X^d + a\eta_1 X^{d-1} + \cdots + a^d\eta_d$. So we obtain from Theorem 3.1.18:

$$\begin{aligned} & R \oplus aI_{[a\alpha]}\zeta_1 \oplus a^2 I_{[a\alpha]}\zeta_2 \oplus \cdots \oplus a^{d-1} I_{[a\alpha]}\zeta_{d-1} \\ = \quad & R[a\alpha] \cap R[(a\alpha)^{-1}] \\ = \quad & R\langle\alpha\rangle \\ = \quad & R \oplus I_{[\alpha]}\zeta_1 \oplus I_{[\alpha]}\zeta_2 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1} \end{aligned}$$

Hence $a^i I_{[a\alpha]} = I_{[\alpha]}$ for each i $(1 \le i \le d-1)$. Noting that $d \ge 3$, we have $aI_{[a\alpha]} = I_{[\alpha]} = a^2 I_{[a\alpha]}$, and hence $I_{[a\alpha]} = aI_{[a\alpha]}$. Since $I_{[a\alpha]}$ is a nonzero, finitely generated ideal of R, we conclude that a must be a unit in R by Lemma 8.2.13. ∎

Proposition 8.2.18 *Let a be a nonzero element in R. Assume that $d \ge 2$ and that both α and $a\alpha$ are anti-integral over R. Assume moreover that there exists $p \in \mathrm{Spec}(R)$ such that $a \in p$ and $I_{[\alpha]} \not\subseteq p$. Then $R[a\alpha] \subsetneq R[\alpha]$.*

Proof Consider R_p. Then α is anti-integral and integral over R_p by Theorem 2.3.2 because $I_{[\alpha]} \not\subseteq p$. Hence if $R[a\alpha] = R[\alpha]$, then $aR_p = R_p$ by Theorem 8.2.14, which is a contradiction because $a \in p$. ∎

Corollary 8.2.19 *Let a be a nonzero element in R. Assume that $d \geq 2$ and that α is ultra-primitive over R. Assume moreover that there exists $p \in \mathrm{Spec}(R)$ such that $a \in p$ and $I_{[\alpha]} \not\subseteq p$. Then $R[a\alpha] \underset{\neq}{\subset} R[\alpha]$.*

Proof This follows from Corollary 8.2.4 and Proposition 8.2.18. ∎

We consider the behaviour of the ring $R\langle a\alpha\rangle$ for $a \in R \setminus (0)$.

Lemma 8.2.20 *Let a, b be elements of R. Assume that a and b are non-zero-divisors on $R/I_{[\alpha]}$. Then $I_{[\alpha]} = I_{[a\alpha]} = I_{[b\alpha]}$ hold. If, in addition, α is a super-primitive element over R, then $aI_{[\alpha]} = bI_{[\alpha]}$ implies that $aR = bR$.*

Proof First we shall show that $I_{[\alpha]} = I_{[a\alpha]} = I_{[b\alpha]}$. Since $\varphi_{a\alpha}(X) = X^d + a\eta_1 X^{d-1} + a^2\eta_2 X^{d-2} + \cdots + a^d\eta_d$ is the minimal polynomial of $a\alpha$ of degree d over K, we have $I_{[\alpha]} \subseteq \bigcap_{i=1}^d I_{a^i\eta_i} = I_{[a\alpha]}$ and $I_{[a\alpha]} = \bigcap_{i=1}^d I_{a^i\eta_i} = \bigcap_{i=1}^d (I_{\eta_i} :_R a^i) \subseteq \bigcap_{i=1}^d (I_{\eta_i} :_R a^d) = (\bigcap_{i=1}^d I_{\eta_i}) :_R a^d = I_{[\alpha]} :_R a^d$. Since a is a non-zero-divisor on $R/I_{[\alpha]}$, it follows that $I_{[\alpha]} = I_{[a\alpha]}$. Next we assume that α is a super-primitive element over R. We have only to show that $b/a \in R$. In order to show that α is super-primitive over R, it suffices that $b/a \in R_p$ for every $p \in \mathrm{Dp}_1(R)$. If $p \not\supseteq I_{[\alpha]}$, then $(I_{[\alpha]})_p = R_p$, so that $aR_p = a(I_{[\alpha]})_p = b(I_{[\alpha]})_p = bR_p$. Hence $b/a \in R_p$. If $p \supseteq I_{[\alpha]}$, then $I_{[\alpha]}R_p = xR_p$ for some $x \in I_{[\alpha]}$ by Theorem 2.3.10 because α is super-primitive. Thus $aI_{[\alpha]}R_p = axR_p = bxR_p = bI_{[\alpha]}R_p$, and hence $aR_p = bR_p$. Therefore $b/a \in R_p$ for all $p \in \mathrm{Dp}_1(R)$, which means that $b/a \in \bigcap_{p \in \mathrm{Dp}_1(R)} R_p = R$ by [Y]. ∎

Lemma 8.2.21 *Let a be an element of R. Assume that a is a non-zero-divisor on $R/I_{[\alpha]}$. Assume that α is a super-primitive element of degree d over R. Then $a\alpha$ is a super-primitive element over R.*

Proof By Lemma 8.2.20, we have

$$J_{[a\alpha]} = I_{[a\alpha]}(1, a\eta_1, \ldots, a^d\eta_d) = I_{[\alpha]}(1, a\eta_1, \ldots, a^d\eta_d)$$

Suppose that $p \supseteq J_{[a\alpha]}$ for some $p \in \mathrm{Dp}_1(R)$. Then $I_{[\alpha]} \subseteq I_{[\alpha]}(1, a\eta_1, \ldots, a^d\eta_d)$. Since α is super-primitive over R, p is a prime divisor of $I_{[\alpha]}$. Hence $a \notin p$ by assumption. Since $a^i\eta_i I_{[\alpha]} \subseteq p$ $(1 \leq i \leq d)$, it holds $\eta_i I_{[\alpha]} \subseteq p$.

Therefore we obtain $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) \subseteq p$. This contradicts that α is super-primitive over R. Hence we see that $p \not\supseteq J_{[a\alpha]}$ for any $p \in \mathrm{Dp}_1(R)$. This yields that $a\alpha$ is super-primitive over R. $\square$

Assume that α is an anti-integral element of degree d over R. Then it is known that

$$R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$$

where $\zeta_i; = \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i$ $(1 \leq i \leq d-1)$, by Theorem 3.1.18. Let $R\langle a\alpha\rangle := R[a\alpha] \cap R[(a\alpha)^{-1}]$ for $a \in R \setminus (0)$, where $R \setminus (0)$ denotes $R \setminus \{0\}$.

Theorem 8.2.22 *Assume that α is a super-primitive element of degree $d(\geq 2)$ over R and let $a, b \in R$. Assume that a and b are non-zero-divisors on $R/I_{[\alpha]}$. Then the following statements are equivalent:*

(1) $R\langle a\alpha\rangle = R\langle b\alpha\rangle$;

(2) $a = ub$ *for some* $u \in U(R)$.

Proof (1)⇒(2): $I_{[\alpha]} = I_{[a\alpha]} = I_{[b\alpha]}$ by Lemma 8.2.20. Since α is super-primitive over R and a, b are non-zero-divisors on $R/I_{[\alpha]}$, it follows that $a\alpha$, $b\alpha$ are super-primitive over R by Lemma 8.2.21, and hence anti-integral over R. Therefore we have the direct sum decompositions of $R\langle a\alpha\rangle$, $R\langle b\alpha\rangle$ as follows:

$$\begin{aligned} R\langle a\alpha\rangle &= R \oplus I_{[a\alpha]}a\zeta_1 \oplus \cdots \oplus I_{[a\alpha]}a^{d-1}\zeta_{d-1} \\ &= R \oplus I_{[\alpha]}a\zeta_1 \oplus \cdots \oplus I_{[\alpha]}a^{d-1}\zeta_{d-1} \\ R\langle b\alpha\rangle &= R \oplus I_{[b\alpha]}a\zeta_1 \oplus \cdots \oplus I_{[b\alpha]}a^{d-1}\zeta_{d-1} \\ &= R \oplus I_{[\alpha]}b\zeta_1 \oplus \cdots \oplus I_{[\alpha]}b^{d-1}\zeta_{d-1} \end{aligned}$$

Note that $\zeta_i = \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i$ $(1 \leq i \leq d-1)$. Since $R\langle a\alpha\rangle = R\langle b\alpha\rangle$ and $d \geq 2$, we conclude $I_{[\alpha]}a\zeta_1 = I_{[\alpha]}b\zeta_1$, and hence $aI_{[\alpha]} = bI_{[\alpha]}$. Therefore $a/b \in R$ by Lemma 8.2.20, and thus $a = ub$ for some $u \in U(R)$.
(2) ⇒ (1): Since $a = ub$ for some $u \in U(R)$, we conclude that $I_{[\alpha]}a^i\zeta_i = I_{[\alpha]}b^i\zeta_i$ for all i $(1 \leq i \leq d-1)$. Hence $R\langle a\alpha\rangle = R\langle b\alpha\rangle$ by the above equalities. $\square$

Corollary 8.2.23 *Let $S := R \setminus \bigcup p_i$ $(p_i \in \mathrm{Ass}R(R/I_{[\alpha]})$. Consider a mapping $\Phi : S \to \{R\langle a\alpha\rangle \mid a \in S\}$ sending $a \in S$ to $R\langle a\alpha\rangle$. Then* $\mathrm{Ker}(\Phi) = U(R)$ *and it induces the bijection $\overline{\Phi} : S/U(R) \cong \{R\langle a\alpha\rangle \mid a \in S\}$.*

Lemma 8.2.24 *Assume that α is an ultra-primitive element over R. Then, for any $a \in R$, $a\alpha$ is ultra-primitive over R, and therefore $a\alpha$ is an anti-integral element over R.*

Proof Since α is ultra-primitive over R, then α is anti-integral over R by Theorem 8.1.1. Hence α is anti-integral over R_p for any $p \in \mathrm{Dp}_1(R)$ by Theorem 2.4.3. If $p \not\supseteq \mathcal{C}(\overline{R}/R)$, then R_p is normal, and hence $a\alpha$ is anti-integral by Theorem 2.2.9. If $p \not\supseteq I_{[\alpha]}$, then α is integral over R_p by Theorem 2.3.2, and thus $a\alpha$ is integral over R_p. Note that $a\alpha$ has the monic relation of degree d. Therefore $a\alpha$ is anti-integral over R_p. ∎

Recall that $\overline{R}$ denotes the integral closure of R in K.

Theorem 8.2.25 *Assume that α is an ultra-primitive element of degree d (≥ 3) over R and that $\overline{R}$ is a finite R-module. Then for elements $a, b \in R \setminus (0)$, the following statements are equivalent:*

(1) $R\langle a\alpha\rangle = R\langle b\alpha\rangle$;

(2) $a = ub$ *for some* $u \in U(R)$.

Proof (2) ⇒ (1): Since α is ultra-primitive over R, $a\alpha$ and $b\alpha$ are anti-integral over R by Lemma 8.2.24. The rest of this implication can be seen in the same way as in the proof of Theorem 8.2.22.
(1) ⇒ (2): Since $a\alpha$ and $b\alpha$ are anti-integral over R by Lemma 8.2.24, we have the decompositions:

$$\begin{aligned} R\langle a\alpha\rangle &= R \oplus I_{[a\alpha]}a\zeta_1 \oplus \cdots \oplus I_{[a\alpha]}a^{d-1}\zeta_{d-1} \\ R\langle b\alpha\rangle &= R \oplus I_{[b\alpha]}b\zeta_1 \oplus \cdots \oplus I_{[b\alpha]}b^{d-1}\zeta_{d-1} \end{aligned}$$

Since $d \geq 3$, we have $aI_{[a\alpha]} = bI_{[b\alpha]}$. It follows that $b^2 I_{[b\alpha]} = a^2 I_{[a\alpha]} = abI_{[b\alpha]}$, and hence $bI_{[b\alpha]} = aI_{[b\alpha]}$. Therefore $a = ub$ for some $u \in U(R)$ by Lemma 8.2.20. ∎

Corollary 8.2.26 *If α is an ultra-primitive element over R, then the mapping $\overline{\Phi} : R\backslash(0)/U(R) \to \{R\langle a\alpha\rangle \mid a \in R\backslash(0)\}$ defined by $\overline{\Phi}(a \bmod U(R)) = R\langle a\alpha\rangle$ for each $a \in R \setminus (0)$, is bijective.*

8.3 Subrings of Type $R[H\alpha] \cap R[(H\alpha)^{-1}]$

In this section, we treat a subring $R[H\alpha] \cap R[(H\alpha)^{-1}]$ with an invertible ideal H.

Let B be an intermediate ring between R and K. Put $I^B_{\eta_i} := B :_B \eta_i$ and $I^B_{[\alpha]} := \bigcap_{i=1}^d I^B_{\eta_i}$, $J^B_{[\alpha]} := I^B_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Note that if B is flat over R, then $I^B_{[\alpha]} = I_{[\alpha]}B$ and $J^B_{[\alpha]} = J_{[\alpha]}B$. In particular, take a nonzero element $t \in R$.

If α is anti-integral over R, then α is also anti-integral over $R[1/t]$. Moreover if $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$, then $\mathrm{grade}(I_{[\alpha]}[1/t] + \mathcal{C}(\overline{R}/R)[1/t]) > 1$, which will be used in this section.

Let H denote an invertible ideal in R. Then for each prime ideal p of R, $H_p = a(p)R_p$ for some $a(p) \in H$. Let the minimal (monic) polynomial of $a(p)\alpha$ be denoted by $\varphi_{a(p)\alpha}(X) := X^d + \eta_1^{(a(p))}X^{d-1} + \cdots + \eta_d^{(a(p))}$. Then $\varphi_{a(p)\alpha}(X) = X^d + a(p)\eta_1 X^{d-1} + \cdots + a(p)^d\eta_d$. So $I_{[a(p)\alpha]} = \bigcap_{i=1}^d I_{\eta_i^{(a(p))}} = \bigcap_{i=1}^d I_{a(p)^i\eta_i}$. Define

$$I_{[H\alpha]} := \bigcap_{p\in \mathrm{Spec}(R)} I_{[a(p)\alpha]}$$

Since H is an invertible ideal of R, H is finitely generated. So there exist $a_k \in H$ $(1 \le k \le \ell)$ and $t_k \in R$ $(1 \le k \le \ell)$ such that $H[1/t_k] = a_k R[1/t_k]$ $(1 \le k \le \ell)$ and that $(t_1, \ldots, t_\ell)R = R$. So we have $I_{[H\alpha]} := \bigcap_{k=1}^\ell I_{[a_k\alpha]}$, and we have $I_{[H\alpha]}^{R[1/t_k]} = I_{[H\alpha]}[1/t_k] = I_{[a_k\alpha]}[1/t_k]$ because $R[1/t_k]$ is flat over R. In this case, $H = (a_1, \ldots, a_\ell)R$. Let $b_1, \ldots, b_m$ be a set of generators of H. Then for each j, there exist i_j such $b_j \in a_{i_j}R[1/t_{i_j}]$. In this case, $b_j = a_{i_j}c_{i_j}/t_{i_j}^{s_{i_j}}$ for some $c_{i_j} \in R$ and $s_{i_j} \in \mathbf{N}$. Since $I_{[a_{i_j}\alpha]}^{R[1/t_{i_j}]} \subseteq I_{[a_{i_j}c_{i_j}\alpha]}^{R[1/t_{i_j}]} \subseteq I_{[b_j\alpha]}^{R[1/t_{i_j}]}$, we have $\bigcap_{k=1}^\ell I_{[a_k\alpha]} \subseteq \bigcap_{j=1}^m I_{[b_j\alpha]}^{R[1/t_{i_j}]} \subseteq \bigcap_{j=1}^m I_{[b_j\alpha]}(R[1/t_{i_j}]) \subseteq \bigcap_{j=1}^m I_{[b_j\alpha]}$. By symmetry, we have $\bigcap_{k=1}^\ell I_{[a_k\alpha]} \supseteq \bigcap_{j=1}^m I_{[b_j\alpha]}$. Hence $I_{[H\alpha]}$ does not depend on a choice of generators of H.

In this section, we fix the generators $a_1, \ldots, a_\ell$ of H and the elements $t_1, \ldots, t_\ell \in R$ with the above properties if necessary.

Put $\varphi_{a_k\alpha}(X) := X^d + \eta_1^{(a_k)}X^{d-1} + \cdots + \eta_d^{(a_k)}$. Then $\varphi_{a_k\alpha}(X) = X^d + a_k\eta_1 X^{d-1} + \cdots + a_k^d\eta_d$. So we have $\eta_i^{(a_k)} = a_k^i\eta_i$ $(1 \le i \le d)$ and $(1 \le k \le \ell)$. Hence $I_{[\alpha]} \subseteq I_{[H\alpha]}$ by definition.

Lemma 8.3.1 *If $H \not\subseteq P$ for every $P \in \mathrm{Ass}_R(R/I_{[\alpha]})$, then $a_k \notin P'$ for every $P' \in \mathrm{Ass}_{R[1/t_k]}(R[1/t_k]/I_{[\alpha]}[1/t_k])$ for every k $(1 \le k \le \ell)$.*

Proof Take $P' \in \mathrm{Ass}_{R[1/t_k]}(R[1/t_k]/I_{[\alpha]}[1/t_k])$ such that $H[1/t_k] \not\subseteq P'$. Note that $P' \cap R \in \mathrm{Ass}_R(R/I_{[\alpha]})$. Thus $H[1/t_k] \not\subseteq P'$ yields $H \not\subseteq P' \cap R$. Since $H[1/t_k] = a_k R[1/t_k]$, we have $a_k \notin P' \cap R$. Hence $a_k \notin P'$. ∎

Remark 8.3.2 Assume that $d = 1$. If α and $a_k\alpha$ $(1 \le k \le \ell)$ are anti-integral over R, then $R[H\alpha] \cap R[(H\alpha)^{-1}] = R\langle\alpha\rangle = R$. ∎

Theorem 8.3.3 *Assume that $d \ge 2$ and that α is ultra-primitive over R. Then $R[H\alpha] \cap R[(H\alpha)^{-1}] \subseteq R\langle\alpha\rangle$ if and only if $HI_{[H\alpha]} \subseteq I_{[\alpha]}$.*

Proof By Corollary 8.2.4, α and $a_k\alpha$ $(1 \leq k \leq \ell)$ are anti-integral over R. We have only to show $R[a_k\alpha][1/t_k] \cap R[(a_k\alpha)^{-1}][1/t_k] \subseteq R[\alpha][1/t_k] \cap R[\alpha^{-1}][1/t_k]$ if and only if $a_k I_{[H\alpha]}[1/t_k] \subseteq I_{[\alpha]}[1/t_k]$ for all k $(1 \leq k \leq \ell)$. Since $R[1/t_k]$ is flat over R, we have $a_k I_{[a_k\alpha]}[1/t_k] = H[1/t_k]I_{[H\alpha]}^{R[1/t_k]}$ and $I_{[\alpha]}[1/t_k] = I_{[\alpha]}^{R[1/t_k]}$. Hence we need only to prove that $R[a_k\alpha][1/t_k] \cap R[(a_k\alpha)^{-1}][1/t_k] \subseteq R[\alpha][1/t_k] \cap R[\alpha^{-1}][1/t_k]$ if and only if $a_k I_{[H\alpha]}^{R[1/t_k]} \subseteq I_{[\alpha]}^{R[1/t_k]}$ for all k $(1 \leq k \leq \ell)$. Since α is anti-integral over R, α is anti-integral over $R[1/t_k]$. Hence replacing $R[1/t_k]$ (respectively a_k) by R (respectively a), we have only to show that $R[a\alpha] \cap R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle$ if and only if $a I_{[a\alpha]} \subseteq I_{[\alpha]}$. This follows Proposition 8.2.6. ∎

Corollary 8.3.4 *Assume that $d \geq 2$ and that R is normal i.e., integrally closed in K. Then $R[H\alpha] \cap R[(H\alpha)^{-1}] \subseteq R\langle\alpha\rangle$ if and only if $H I_{[H\alpha]} \subseteq I_{[\alpha]}$.*

Proof Since R is a Noetherian normal domain, α is anti-integral over R and $a_k\alpha$ is anti-integral over $R[1/t_k]$ for each k by Theorem 2.2.9. So the conclusion follows from Proposition 8.2.6. ∎

Corollary 8.3.5 *Assume that α is super-primitive over R and that $H \not\subseteq P$ for every $P \in \mathrm{Ass}_R(R/I_{[\alpha]})$. Then $R[H\alpha] \cap R[(H\alpha)^{-1}] \subseteq R\langle\alpha\rangle$ if and only if $H I_{[H\alpha]} \subseteq I_{[\alpha]}$.*

Proof Note first that $a_k\alpha$ is super-primitive over R by Lemma 8.3.1 and Proposition 8.2.2. Since a super-primitive element is an anti-integral element by Theorem 2.2.8, the conclusion follows from Proposition 8.2.2. ∎

Theorem 8.3.6 *Assume that α is ultra-primitive over R. Assume that $H \not\subseteq P$ for every $P \in \bigcup_{i=1}^{d} \mathrm{Ass}_R(R/I_{\eta_i})$. Then $R[H\alpha] \cap R[(H\alpha)^{-1}] \subseteq R\langle\alpha\rangle$.*

Proof By Corollary 8.2.4, α and $a_k\alpha$ $(1 \leq k \leq \ell)$ are anti-integral over R. We have only to show $R[a_k\alpha][1/t_k] \cap R[(a_k\alpha)^{-1}][1/t_k] \subseteq R[\alpha][1/t_k] \cap R[\alpha^{-1}][1/t_k]$ for each k $(1 \leq k \leq \ell)$. Note that a_k is not a zero-divisor on $R/I\eta_i$ for all k and i. We have only to show that $R[a\alpha] \cap R[(a\alpha)^{-1}] \subseteq R\langle\alpha\rangle$ by replacing $R[1/t_k]$ (resp. a_k) by R (resp. a). This follows from Proposition 8.2.8. ∎

Corollary 8.3.7 *Assume that R is normal. Assume that $H \not\subseteq P$ for each $P \in \bigcup_{i=1}^{d} \mathrm{Ass}_R(R/I_{\eta_i})$. Then $R[H\alpha] \cap R[(H\alpha)^{-1}] \subseteq R\langle\alpha\rangle$.*

Proof Both α and $a_k\alpha$ are anti-integral over R by Theorem 2.2.9. Our conclusion follows from Proposition 8.2.8. ∎

Corollary 8.3.8 *Assume that α is super-primitive over R and that $H \not\subseteq P$ for every $P \in \mathrm{Ass}_R(R/I_{[\alpha]})$. Then $R[H\alpha] \cap R[(H\alpha)^{-1}] \subseteq R\langle\alpha\rangle$.*

Proof Note that $a_k\alpha$ is super-primitive over R by Lemma 8.3.1 and Proposition 8.2.2. Since a super-primitive element is an anti-integral element by Theorem 2.2.8, the conclusion follows from Proposition 8.2.8. ∎

In the rest of this section, we examine some conditions under which the equality $R[H\alpha] \cap R[(H\alpha)^{-1}] = R\langle\alpha\rangle$ holds.

Theorem 8.3.9 *Assume that $d \geq 2$ and that α is ultra-primitive over R. Assume moreover that α is integral over R. Then the following statements are equivalent:*

(1) $R[H\alpha] \cap R[(H\alpha)^{-1}] = R\langle\alpha\rangle$;

(2) $R[H\alpha] = R[\alpha]$;

(3) $H^i I_{[H\alpha]} = I_{[\alpha]}$ *for all* i $(1 \leq i \leq d-1)$;

(4) $H = R$.

Proof By Corollary 8.2.4, α and $a_k\alpha$ $(1 \leq k \leq \ell)$ are anti-integral over R. By the choice of a_k and t_k $(1 \leq k \leq \ell)$, we have the following equivalences:
(1) $\Leftrightarrow$ $R[a_k\alpha][1/t_k] \cap R[(a_k\alpha)^{-1}][1/t_k] = R[\alpha][1/t_k] \cap R[\alpha^{-1}][1/t_k]$ for all k;
(2) $\Leftrightarrow$ $R[a_k\alpha][1/t_k] = R[\alpha][1/t_k]$ for all k;
(3) $\Leftrightarrow$ $a_k^i I_{[H\alpha]}[1/t_k] = I_{[\alpha]}[1/t_k]$ for all k and for all i $(1 \leq k \leq \ell, 1 \leq i \leq d-1)$ $\Leftrightarrow$ $a_k^i I_{[a_k\alpha]}^{R[1/t_k]} = I_{[\alpha]}^{R[1/t_k]}$ for all k and for all i $(1 \leq k \leq \ell, 1 \leq i \leq d-1)$;
(4) $\Leftrightarrow$ $a_k R[1/t_k] = R[1/t_k]$ for all k. By the same argument as in the proof of Proposition 8.2.6, replacing $R[1/t_k]$ (resp. a_k) by R (resp. a) if necessary, we need only to show the following equivalences among (1′), (2′), (3′), and (4′):
(1′) $R[a\alpha] \cap R[(a\alpha)^{-1}] = R\langle\alpha\rangle$;
(2′) $R[a\alpha] = R[\alpha]$ and $R[(a\alpha)^{-1}] = R[\alpha^{-1}]$;
(3′) $a^i I_{[a\alpha]} = I_{[\alpha]}$ for all i $(1 \leq i \leq d-1)$;
(4′) $aR = R$, that is, a is a unit in R.
These equivalences follow from Theorem 8.2.14. ∎

Corollary 8.3.10 *Assume that R is normal. Assume further that $d \geq 2$ and that α is integral over R. Then the following statements are equivalent:*

(1) $R[H\alpha] \cap R[(H\alpha)^{-1}] = R\langle\alpha\rangle$;

(2) $R[H\alpha] = R[\alpha]$;

(3) $H^i I_{[H\alpha]} = I_{[\alpha]}$ *for all* i $(1 \leq i \leq d-1)$;

(4) $H = R$.

Proof Since R is a Noetherian normal domain, α is anti-integral over R and $a_k\alpha$ is anti-integral over $R[1/t_k]$ for each k by Theorem 2.2.9. Our conclusion follows from Theorem 8.2.14. ∎

Corollary 8.3.11 *Assume that α is super-primitive over R and that $H \not\subseteq P$ for every $P \in \mathrm{Ass}_R(R/I_{[\alpha]})$. Assume further that $d \geq 2$ and that α is integral over R. Then the following statements are equivalent:*

(1) $R[H\alpha] \cap R[(H\alpha)^{-1}] = R\langle\alpha\rangle$;

(2) $R[H\alpha] = R[\alpha]$;

(3) $H^i I_{[H\alpha]} = I_{[\alpha]}$ *for all* i $(1 \leq i \leq d-1)$;

(4) $H = R$.

Proof Note that $a_k\alpha$ is super-primitive over R by Lemma 8.3.1 and Proposition 8.2.2. Since a super-primitive element is an anti-integral element by Theorem 2.2.8, the conclusion follows from Theorem 8.2.14. ∎

Theorem 8.3.12 *Assume that α is ultra-primitive over R. Assume moreover that $d \geq 3$. Then the following statements are equivalent:*

(1) $R[H\alpha] \cap R[(H\alpha)^{-1}] = R\langle\alpha\rangle$;

(2) $R[H\alpha] = R[\alpha]$ *and* $R[(H\alpha)^{-1}] = R[\alpha^{-1}]$;

(3) $H^i I_{[H\alpha]} = I_{[\alpha]}$ *for all* i $(1 \leq i \leq d-1)$;

(4) $H = R$.

Proof By Corollary 8.2.4, α and $a_k\alpha$ $(1 \leq k \leq \ell)$ are anti-integral over R. By the choice of a_k and t_k $(1 \leq k \leq \ell)$, we have the following equivalences:
(1) $\Leftrightarrow$ $R[a_k\alpha][1/t_k] \cap R[(a_k\alpha)^{-1}][1/t_k] = R[\alpha][1/t_k] \cap R[\alpha^{-1}][1/t_k]$ for all k;
(2) $\Leftrightarrow$ $R[a_k\alpha][1/t_k] = R[\alpha][1/t_k]$ and $R[(a_k\alpha)^{-1}][1/t_k] = R[\alpha^{-1}][1/t_k]$ for all k;
(3) $\Leftrightarrow a_k^i I_{[H\alpha]}[1/t_k] = I_{[\alpha]}[1/t_k]$ for all k and for all i $(1 \leq k \leq \ell, 1 \leq i \leq d-1)$
$\Leftrightarrow a_k^i I_{[a_k\alpha]}^{R[1/t_k]} = I_{[\alpha]}^{R[1/t_k]}$ for all k and for all i $(1 \leq k \leq \ell, 1 \leq i \leq d-1)$;
(4) $\Leftrightarrow a_k R[1/t_k] = R[1/t_k]$ for all k.
By the same argument as in the proof of Theorem 8.3.9, replacing $R[1/t_k]$ (resp. a_k) by R (resp. a) if necessary, we need only to show the following equivalences among (1'), (2'), (3'), and (4'):

(1') $R[a\alpha] \cap R[(a\alpha)^{-1}] = R\langle\alpha\rangle$;
(2') $R[a\alpha] = R[\alpha]$ and $R[(a\alpha)^{-1}] = R[\alpha^{-1}]$;
(3') $a^i I_{[a\alpha]} = I_{[\alpha]}$ for all i $(1 \leq i \leq d-1)$;
(4') $aR = R$, that is, a is a unit in R.
These equivalences follow from Theorem 8.2.16. ∎

Corollary 8.3.13 *Assume that R is normal. Assume moreover that $d \geq 3$. Then the following statements are equivalent:*

(1) $R[H\alpha] \cap R[(H\alpha)^{-1}] = R\langle\alpha\rangle$;
(2) $R[H\alpha] = R[\alpha]$ *and* $R[(H\alpha)^{-1}] = R[\alpha^{-1}]$;
(3) $H^i I_{[H\alpha]} = I_{[\alpha]}$ *for all* i $(1 \leq i \leq d-1)$;
(4) $H = R$.

Proof Both α and $a_k\alpha$ are anti-integral over R by Theorem 2.2.9. Our conclusion follows from Theorem 8.2.16. ∎

Corollary 8.3.14 *Assume that α is super-primitive over R and that $H \not\subseteq P$ for every $P \in \mathrm{Ass}_R(R/I_{[\alpha]})$. Assume further that $d \geq 3$. Then the following statements are equivalent:*

(1) $R[H\alpha] \cap R[(H\alpha)^{-1}] = R\langle\alpha\rangle$;
(2) $R[H\alpha] = R[\alpha]$ *and* $R[(H\alpha)^{-1}] = R[\alpha^{-1}]$;
(3) $H^i I_{[H\alpha]} = I_{[\alpha]}$ *for all* i $(1 \leq i \leq d-1)$;
(4) $H = R$.

Proof Note that $a_k\alpha$ is super-primitive over R by Lemma 8.3.1 and Proposition 8.2.2. Since a super-primitive element is an anti-integral element by Theorem 8.2.8, the conclusion follows from Theorem 8.2.16. ∎

Theorem 8.3.15 *Assume that $d \geq 2$ and that α is ultra-primitive over R. Assume moreover that there exists $p \in \mathrm{Spec}(R)$ such that $H \subseteq p$ and $I_{[\alpha]} \not\subseteq p$. Then $R[H\alpha] \subsetneq R[\alpha]$.*

Proof By Corollary 8.2.4, α and $a_k\alpha$ $(1 \leq k \leq \ell)$ are anti-integral over R. Consider R_p. Then α is anti-integral and integral over R_p because $I_{[\alpha]} \not\subseteq p$. Hence if $R[H\alpha] = R[\alpha]$, then $HR_p = R_p$ by Proposition 8.2.14, which is a contradiction because $H \subseteq p$. ∎

Corollary 8.3.16 *Assume that R is normal. Assume moreover that there exists $p \in \mathrm{Spec}(R)$ such that $H \subseteq p$ and $I_{[\alpha]} \not\subseteq p$. Then $R[H\alpha] \subsetneq R[\alpha]$.*

Proof Since R is a Noetherian normal domain, α is anti-integral over R and $a_k\alpha$ is anti-integral over $R[1/t_k]$ for each k by Theorem 2.2.9. Our conclusion follows from Proposition 8.2.18. ∎

Corollary 8.3.17 *Assume that α is super-primitive over R and that $H \not\subseteq P$ for every $P \in \mathrm{Ass}_R(R/I_{[\alpha]})$. Assume further that there exists $p \in \mathrm{Spec}(R)$ such that $H \subseteq p$ and $I_{[\alpha]} \not\subseteq p$. Then $R[H\alpha] \underset{\neq}{\subset} R[\alpha]$.*

Proof Note that $a_k\alpha$ is super-primitive over R by Lemma 8.3.1 and Proposition 8.2.2. Since a super-primitive element is an anti-integral element by Theorem 2.2.8, the conclusion follows from Proposition 8.2.18. ∎

8.4 A Linear Generator of an Ultra-Primitive Extension $R[\alpha]$

In this section, we investigate some conditions for an element $\lambda\alpha + \mu$ to be a generator of a simple ring extension $R[\alpha]$, where $\lambda, \mu \in K$ and α is an ultra-primitive element over R. Let $R\langle\alpha\rangle := R[\alpha] \cap R[\alpha^{-1}]$. More definitely, our objective is to show the following result: Assume that α is ultra-primitive of degree $d \geq 3$ over R, that α is also ultra-primitive over $R\langle\alpha\rangle$, and that α is exclusive over R, i.e., $R[\alpha] \cap K = R$. Then for $\lambda, \mu \in K$, $R[\alpha] = R[\lambda\alpha + \mu]$ if and only if $\lambda \in U(R)$ and $\mu \in R$, where $U(R)$ denotes the set of the units in R.

An element $\alpha \in L$ is called to be an *ultra-primitive element* over R if α satisfies

$$\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$$

We say that an extension $R[\alpha]$ is an ultra-primitive extension of R if α is ultra-primitive over R. Note that $\mathcal{C}(\overline{R}/R)$ may be (0). In this case, if $\alpha \in K$ is ultra-primitive over R, then the condition $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$ implies that $I_{[\alpha]} = R$ because $\mathrm{grade}(I_{[\alpha]}) = 1$ or $I_{[\alpha]} = R$, and hence that $\alpha \in R$. Namely, if $\mathcal{C}(\overline{R}/R) = (0)$, an ultra-primitive extension of R is R itself.

Remark 8.4.1 Assume that R is normal (i.e., integrally closed in K). Then any algebraic element over R is ultra-primitive over R by definition. ∎

Remark 8.4.2 If α is anti-integral and integral over R, then $R[\alpha]$ is a free R-module $R \oplus R\alpha \oplus \cdots \oplus R\alpha^{d-1}$ (*cf.* Theorem 2.3.8 and Corollary 2.3.5). ∎

Proposition 8.4.3 *Let a be an element in R. Assume that α is an ultra-primitive element of degree $d \geq 3$ over R and that α is also ultra-primitive over $R\langle\alpha\rangle$. Then $R[\alpha] = R[a\alpha]$ if and only if $a \in U(R)$.*

Proof This is seen in Theorem 8.2.17. ∎

Example 8.4.4 For $d \leq 2$, the conclusion of Proposition 8.4.3 is not necessarily valid.

(1) We construct an example in the case $d = 1$.
Let k be the field of the rational numbers $\mathbf{Q}$ and R be a polynomial ring $k[t]$. Let $\alpha := 1/t^2$ and $a := t$. Then $a\alpha = 1/t$, $\varphi_\alpha(X) = X - 1/t^2$, and $\varphi_{a\alpha}(X) = X - 1/t$. Hence $I_{[\alpha]} = t^2R$ and $I_{[a\alpha]} = tR$. We have $R[\alpha] = k[t, 1/t^2] = k[t, 1/t]$, and $R[a\alpha] = k[t, 1/t]$, which means that $R[\alpha] = R[a\alpha]$ but $a \notin U(R)$.

(2) We construct an example in the case $d = 2$.
Let k be the field of the rational numbers $\mathbf{Q}$ and R be a ring $k[s^2, t]$, which is a subring of a polynomial ring $k[s, t]$. Let $\alpha := 1/(st^2)$ and $a := t$. Then $a\alpha = 1/(st)$. We have $\varphi_\alpha(X) = X^2 - (1/(st^2))^2$ and $\varphi_{a\alpha}(X) = X^2 - 1/(st)^2$. In this case, $I_{[\alpha]} = s^2t^4R$ and $I_{[a\alpha]} = s^2t^2R$, and hence $I_{[\alpha]} = t^2I_{[a\alpha]}$. Note that $R[\alpha] = k[s^2, t, 1/(st^2)] = k[s^2, t, 1/s, 1/(st^2)] = k[s, t, 1/s, 1/(st^2)] = k[s, t, 1/s, 1/t]$ and that $R[a\alpha] = k[s^2, t, 1/(st)] = k[s, t, 1/s, 1/t]$, which shows that $R[\alpha] = R[a\alpha]$. But $I_{[\alpha]} = s^2t^4R \neq s^2t^2R = I_{[a\alpha]}$ and hence $I_{[\alpha]} = t^2I_{[a\alpha]}$. It is clear that $a = t \notin U(R)$.

Recall that $\alpha \in L$ is said to be exclusive over R if $R[\alpha] \cap K = R$.

Proposition 8.4.5 *Assume that α is an ultra-primitive element of degree $d \geq 3$ over R, that α is also ultra-primitive over $R\langle\alpha\rangle$, and that α is exclusive over R. Let $\lambda \in R$ and let $\mu \in K$. Then $R[\alpha] = R[\lambda\alpha + \mu]$ if and only if $\lambda \in U(R)$ and $\mu \in R$.*

Proof The implication ($\Leftarrow$) is obvious. Since $\lambda\alpha + \mu \in R[\alpha]$, we have $\mu \in R[\alpha] \cap K = R$. Thus $R[\alpha] = R[\lambda\alpha]$ and hence $\lambda \in U(R)$ by Proposition 8.4.3. ∎

Let $I_\lambda^{\overline{R}}$ denote the ideal $\overline{R} :_{\overline{R}} \lambda$ for $\lambda \in K$ and $I_{[\alpha]}^{\overline{R}}$ denote $\bigcap_{i=1}^{d} I_{\eta_i}^{\overline{R}}$.

Lemma 8.4.6 (1) *Let $\lambda \in K \setminus (0)$. If $\sqrt{I_\lambda^{\overline{R}}} = \sqrt{I_{\lambda^{-1}}^{\overline{R}}}$, then $\lambda \in U(\overline{R})$.*
(2) *Assume that α is an ultra-primitive element of degree $d \geq 3$ over R, that α is*

also ultra-primitive over $R\langle\alpha\rangle$, and that α is exclusive over R. If $R[\alpha] = R[\lambda\alpha]$ for $\lambda \in K \setminus (0)$, then $\lambda \in \overline{R}$.

Proof (1) Since R is Noetherian, $\overline{R}$ is a Krull domain (*cf.* [M1]). Take $P \in \mathrm{Ht}_1(\overline{R})$. Then $\overline{R}_P$ is a DVR. Assume that $\lambda \notin \overline{R}_P$. Then $1/\lambda \in \overline{R}_P$ and hence $(I_\lambda^{\overline{R}})_P = (I_{\lambda^{-1}}^{\overline{R}})_P = \overline{R}_P$. Next assume that $\lambda \in \overline{R}_P$. Then $(I_\lambda^{\overline{R}})_P = \overline{R}_P$. In either case, since $(\sqrt{I_\lambda^{\overline{R}}})_P = (\sqrt{I_{\lambda^{-1}}^{\overline{R}}})_P = \overline{R}_P$, it follows that $(I_\lambda^{\overline{R}})_P = (I_{\lambda^{-1}}^{\overline{R}})_P = \overline{R}_P$. Since $I_\lambda^{\overline{R}}$ and $I_{\lambda^{-1}}^{\overline{R}}$ are divisorial ideals of $\overline{R}$, $I_\lambda^{\overline{R}} = I_{\lambda^{-1}}^{\overline{R}} = \overline{R}$, and hence λ, $\lambda^{-1} \in \overline{R}$. Thus $\lambda \in U(\overline{R})$.
(2) We assume that $I_\lambda^{\overline{R}} \neq \overline{R}$. Note that α is ultra-primitive over $\overline{R}$ by definition because $\mathcal{C}(\overline{R}/\overline{R}) = \overline{R}$. We shall show that $\sqrt{I_\lambda^{\overline{R}}} = \sqrt{I_{\lambda^{-1}}^{\overline{R}}}$. Take $P \in \mathrm{Dp}_1(\overline{R})$ such that $I_\lambda^{\overline{R}} \not\subseteq P$. Then $\lambda \in \overline{R}_P$ and $\overline{R}_P[\alpha] = \overline{R}_P[\lambda\alpha]$, which is induced from $R[\alpha] = R[\lambda\alpha]$. Since α is ultra-primitive over $\overline{R}$, we have $\lambda \in U(\overline{R}_P)$ by Proposition 8.4.5. Thus $\lambda^{-1} \in \overline{R}_P$. Hence $I_{\lambda^{-1}}^{\overline{R}} \not\subseteq P$. Therefore noting that $I_{\lambda^{-1}}^{\overline{R}}$ and $I_\lambda^{\overline{R}}$ are divisorial ideals, we have $\sqrt{I_\lambda^{\overline{R}}} \subseteq \sqrt{I_{\lambda^{-1}}^{\overline{R}}}$. Conversely, take $P \in \mathrm{Dp}_1(\overline{R})$ such that $I_{\lambda^{-1}}^{\overline{R}} \not\subseteq P$. Then $\lambda^{-1} \in \overline{R}_P$. So we have $\overline{R}_P[\alpha] = \overline{R}_P[\lambda^{-1}\alpha]$. Thus $\lambda^{-1} \in U(\overline{R}_P)$ by Proposition 8.4.5. Thus $\lambda \in \overline{R}_P$, that is, $I_\lambda^{\overline{R}} \not\subseteq P$.

Therefore noting that $I_{\lambda^{-1}}^{\overline{R}}$ and $I_\lambda^{\overline{R}}$ are divisorial ideals, we have $\sqrt{I_{\lambda^{-1}}^{\overline{R}}} \subseteq \sqrt{I_\lambda^{\overline{R}}}$. Therefore $\sqrt{I_\lambda^{\overline{R}}} = \sqrt{I_{\lambda^{-1}}^{\overline{R}}}$. The conclusion follows from (1). ∎

Proposition 8.4.7 *Assume that α is an ultra-primitive element of degree $d \geq 3$ over R, that α is also ultra-primitive over $R\langle\alpha\rangle$, and that α is exclusive over R. Then for $\lambda \in K$ and $\mu \in R$, $R[\alpha] = R[\lambda\alpha + \mu]$ if and only if $\lambda \in U(R)$.*

Proof Since $\mu \in R$, we may assume that $\mu = 0$, that is, $R[\alpha] = R[\lambda\alpha]$. The implication ($\Leftarrow$) is obvious. We shall show the converse implication.

Case I: When $\mathcal{C}(\overline{R}/R) = (0)$, $I_{[\alpha]} = R$ by definition of grade and the assumption. Thus $R[\alpha] = R[\lambda\alpha]$ is a free R-module $R \oplus R\alpha \oplus \cdots \oplus R\alpha^{d-1}$. Since $1, \alpha, \ldots, \alpha^{d-1}$ are linearly independent over K, we have $\lambda\alpha = b\alpha$ for some $b \in R$. Thus $\lambda = b \in R$ and hence $\lambda \in U(R)$ by Proposition 8.4.5. We are done.

Case II: Now consider the case that $\mathcal{C}(\overline{R}/R) \neq (0)$.

Step 1: When $\lambda \in R$, i.e., $I_\lambda = R$, $\lambda \in R$ and hence $\lambda \in U(R)$ by Proposition 8.4.5.

Step 2: Suppose that $\lambda \notin R$, i.e., $I_\lambda \neq R$. By Lemma 8.4.6(2), we have $\lambda \in \overline{R}$.

Let p be a prime divisor of $I_\lambda(\neq (0))$. Then $p \in \mathrm{Dp}_1(R)$ and $0 \neq \mathcal{C}(\overline{R}/R) \subseteq p$ because $\lambda \in \overline{R}$. So $I_{[\alpha]} \not\subseteq p$ since α is ultra-primitive over R. Thus $R_p[\alpha]$ is a free R_p-module $R_p \oplus R_p\alpha \oplus \cdots \oplus R_p\alpha^{d-1}$. Note that $R_p[\lambda\alpha]$ is a finitely generated R_p-module and is generated by $1, \lambda\alpha, \ldots, (\lambda\alpha)^{n-1}$ for some $n \in \mathbf{N}$. Thus $R_p[\lambda\alpha] = R_p \oplus R_p(\lambda\alpha) \oplus \cdots \oplus R_p(\lambda\alpha)^{n-1}$. Since $R_p[\alpha] = R_p[\lambda\alpha]$ and they are free R_p-modules of rank d and $1, \lambda\alpha, \ldots, (\lambda\alpha)^{d-1}$ are linearly independent over K. So we have $d = n$. Hence $R_p\alpha = R_p(\lambda\alpha)$; $R_p = R_p\lambda$, so that $\lambda \in R_p$, that is, $I_\lambda R_p = R_p$, which contradicts the assumption $I_\lambda \subseteq p$. Therefore $\lambda \in R$. By Proposition 8.4.5, we conclude that $\lambda \in U(R)$. This completes the proof. $\square$

Proposition 8.4.8 *Let β be an element in $R[\alpha] = R + R\alpha + \cdots + R\alpha^{d-1}$. Then β has a monic algebraic relation of degree d over R.*

Proof Let $\beta\alpha^i = \sum \lambda_{i+1,j+1}\alpha^j$ $(0 \leq i \leq d-1, 0 \leq j \leq d-1)$ with $\lambda_{i,j} \in R$. In this case, $\det(\beta\delta_{ij} - \lambda_{i,j}) = 0$, which is a monic algebraic relation in β. Hence β has a monic algebraic relation of degree d over R. $\square$

Lemma 8.4.9 *Let $\lambda \in K \setminus (0)$. Assume that* $\mathrm{grade}(I_\lambda + I_{\lambda^{-1}}) = 1$. *Then $\mathcal{C}(\overline{R}/R) \subseteq \sqrt{I_\lambda}$.*

Proof If $\mathcal{C}(\overline{R}/R) = (0)$, then $\mathcal{C}(\overline{R}/R) \subseteq \sqrt{I_\lambda}$. So we assume that $\mathcal{C}(\overline{R}/R) \neq (0)$ in the sequel.

Step 1: Take $p \in \mathrm{Dp}_1(R)$. If $p \supseteq I_\lambda$, then $p \supseteq I_{\lambda^{-1}}$. Indeed, If $p \not\supseteq I_{\lambda^{-1}}$, then $\mathrm{grade}(I_\lambda + I_{\lambda^{-1}}) \geq 2$, a contradiction. Since $\mathrm{grade}(I_\lambda) = 1$, we have $\sqrt{I_{\lambda^{-1}}} \subseteq \sqrt{I_\lambda}$. By symmetry, we also have $\sqrt{I_{\lambda^{-1}}} \supseteq \sqrt{I_\lambda}$. Thus $\sqrt{I_{\lambda^{-1}}} = \sqrt{I_\lambda}$.

Step 2: We shall show that if $\mathcal{C}(\overline{R}/R) \not\subseteq \sqrt{I_\lambda}$, then $\mathrm{grade}(I_\lambda^{\overline{R}} + I_{\lambda^{-1}}^{\overline{R}}) = 1$. Suppose that $\mathrm{grade}(I_\lambda^{\overline{R}} + I_{\lambda^{-1}}^{\overline{R}}) \geq 2$ and $\mathcal{C}(\overline{R}/R) \not\subseteq \sqrt{I_\lambda}$. Let q be a prime divisor of $\sqrt{I_{\lambda^{-1}}} = \sqrt{I_\lambda}$ such that $\mathcal{C}(\overline{R}/R) \not\subseteq q$. Then $\mathrm{grade}(q) = 1$. Note that $(\sqrt{I_\lambda})\overline{R} = (\sqrt{I_{\lambda^{-1}}})\overline{R}$. By Lying-Over Theorem, $q\overline{R} \neq \overline{R}$. Moreover $\mathcal{C}(\overline{R}/R) = \mathcal{C}(\overline{R}/R)\overline{R} \not\subseteq q\overline{R}$. So there exists a prime ideal P' of $\overline{R}$ lying over q such that $\mathcal{C}(\overline{R}/R) \not\subseteq P'$. Take a nonzero element $a \in (I_\lambda^{\overline{R}} \cap R) \cap q$. Then there exists a prime divisor P of $a\overline{R}$ such that $P \subseteq P'$. In this case, $P \cap R \subseteq P' \cap R = q$. Let $\{P_1, \ldots, P_\ell\}$ be the set of all prime divisors of $a\overline{R}$. Let $P := P_1$. Note that $P_i \in \mathrm{Ht}_1(\overline{R})$ for all i $(1 \leq i \leq \ell)$. Since $\mathrm{grade}(I_\lambda^{\overline{R}} + I_{\lambda^{-1}}^{\overline{R}}) \geq 2$ and $I_\lambda \subseteq I_\lambda^{\overline{R}}$, there exists $\beta \in I_{\lambda^{-1}}^{\overline{R}}$ such that a, β form a regular sequence in $\overline{R}$. Note here that $\beta \notin P_i$ for all i. Since β is integral over R, there is an integral relation: $\beta^s + a_1\beta^{s-1} + \cdots + a_s = 0$ with $a_i \in R$ $(a_s \neq 0)$. Since $I_{\lambda^{-1}}^{\overline{R}} \not\subseteq \bigcup_{i=1}^{\ell} P_i$, there exists $e \in I_{\lambda^{-1}}^{\overline{R}} \cap R$ such that $e \notin \bigcup_{i=1}^{\ell} P_i$. Consider $e\beta$ instead of β, we can

assume that $a_i \in I_{\lambda^{-1}}^{\overline{R}} \cap R$ for all $(1 \leq i \leq s)$. Let $\{a_{t_1}, a_{t_2}, \ldots, a_{t_k}\}$ $(t_1 < t_2 < \cdots < t_k = s)$ be the set of the nonzero coefficients of $\beta^s + a_1\beta^{s-1} + \cdots + a_s = 0$. If $a_s \notin \bigcup_{i=1}^{\ell} P_i$, then a, a_s form a regular sequence in $\overline{R}$ with $a \in I_{\lambda}^{\overline{R}}$, $a_s \in I_{\lambda^{-1}}^{\overline{R}}$. If $a_s \in \bigcup P_i$, then $\beta^{t_{k-1}} + a_1\beta^{t_{k-1}-1} + \cdots + a_{t_{k-1}} \in \bigcup P_i$. If $a_{t_{k-1}} \notin \bigcup P_i$, then $a, a_{t_{k-1}}$ form a regular sequence in $\overline{R}$ with $a \in I_{\lambda}^{\overline{R}}$, $a_{t_{k-1}} \in I_{\lambda^{-1}}^{\overline{R}}$. If $a_{t_{k-1}} \in \bigcup P_i$, then $\beta^{t_{k-2}} a_1\beta^{t_{k-2}-1} + \cdots + a_{t_{k-2}} \in \bigcup P_i$. Continuing this process, we find a_{t_j} such that $a_{t_j} \notin \bigcup P_i$ (and hence $a_{t_j} \notin q$ because of grade$(q) = 1$) because $\beta \notin \bigcup P_i$. In this case, a, a_{t_j} form a regular sequence in $\overline{R}$ with $a \in I_{\lambda}^{\overline{R}} \cap R$, $a_{t_j} \in I_{\lambda^{-1}}^{\overline{R}} \cap R$. Since $\mathcal{C}(\overline{R}/R) \not\subseteq q$, we can choose an element $c \in \mathcal{C}(\overline{R}/R) \backslash q$. By construction, $a_{t_j} \in I_{\lambda^{-1}}^{\overline{R}} \cap R$, that is, $a_{t_j}\lambda^{-1} \in \overline{R}$. Put $b := a_{t_j}c$. Then $b\lambda^{-1} = c(a_{t_j}\lambda^{-1}) \in R$. Hence $b \in R :_R \lambda^{-1} = I_{\lambda^{-1}} \subseteq q$. But since $c, a_{t_j} \notin q$, we have $b = ca_{t_j} \notin q$, a contradiction. Therefore if $\mathcal{C}(\overline{R}/R) \not\subseteq \sqrt{I_\lambda}$, then grade$(I_{\lambda}^{\overline{R}} + I_{\lambda^{-1}}^{\overline{R}}) = 1$.

Step 3: We conclude that if grade$(I_\lambda + I_{\lambda^{-1}}) = 1$, then $\mathcal{C}(\overline{R}/R) \subseteq \sqrt{I_\lambda}$ or grade$(I_{\lambda}^{\overline{R}} + I_{\lambda^{-1}}^{\overline{R}}) = 1$.

Step 4: Assume that grade$(I_{\lambda}^{\overline{R}} + I_{\lambda^{-1}}^{\overline{R}}) = 1$. In this case, we have $\sqrt{I_{\lambda^{-1}}^{\overline{R}}} = \sqrt{I_{\lambda}^{\overline{R}}}$ by the same argument as in (Step 1). Hence $\lambda \in U(\overline{R})$ by Lemma 8.4.6 (1). This means that $\lambda\mathcal{C}(\overline{R}/R) \subseteq R$ and hence that $\mathcal{C}(\overline{R}/R) \subseteq I_\lambda$.

These steps complete the proof. ∎

Theorem 8.4.10 *Assume that α is an ultra-primitive element of degree $d \geq 3$ over R, that α is also ultra-primitive over $R\langle\alpha\rangle$, and that α is exclusive over R, i.e., $R[\alpha] \cap K = R$. Then for $\lambda, \mu \in K$, $R[\alpha] = R[\lambda\alpha + \mu]$ if and only if $\lambda \in U(R)$ and $\mu \in R$.*

Proof ($\Leftarrow$) is obvious.
($\Rightarrow$): When $\mathcal{C}(\overline{R}/R) = (0)$, $I_{[\alpha]} = R$ by definition of grade and the assumption. So $R[\alpha]$ is a free R-module $R \oplus R\alpha \oplus \cdots \oplus R\alpha^{d-1}$ by Remark 8.4.2. Since $R[\alpha] = R[\lambda\alpha + \mu]$, we have $\lambda\alpha + \mu = b_0 + b_1\alpha + \cdots + b_{d-1}\alpha^{d-1}$ for some $b_i \in R$. Since $1, \alpha, \ldots, \alpha^{d-1}$ are linearly independent over K. Hence $\lambda = b_1 \in R$ and $\mu = b_0 \in R$. Thus $\lambda \in U(R)$ by Proposition 8.4.7 and $\mu \in R$. We are done.

Now we consider the case $\mathcal{C}(\overline{R}/R) \neq (0)$. We divide a proof into three steps.

Step 1: First, we show that $\sqrt{I_\lambda} = \sqrt{I_\mu}$. Take a prime ideal p of R. Assume that $I_\lambda \not\subseteq p$. Then $\lambda \in R_p$. So we have $\mu \in R_p[\alpha] \cap K$. Since α is exclusive, $R_p[\alpha] \cap K = R_p$. Hence $\mu \in R_p$, which means that $I_\mu \not\subseteq p$. Conversely, assume that $I_\mu \not\subseteq p$. Then $\mu \in R_p$. So we have $\lambda \in R_p$ by Proposition 8.4.7, which yields that $I_\lambda \not\subseteq p$. Therefore $\sqrt{I_\lambda} = \sqrt{I_\mu}$.

Step 2: Next, take a prime ideal p of R. Assume that $\sqrt{I_\lambda} = \sqrt{I_\mu} \not\subseteq p$. Then $\lambda, \mu \in R_p$, and hence $\lambda \in U(R_p)$ by Proposition 8.4.5. Thus $I_{\lambda^{-1}} \not\subseteq p$. Therefore $\sqrt{I_{\lambda^{-1}}} \supseteq \sqrt{I_\lambda}$.

Step 3: Now we shall prove $I_{\lambda^{-1}} = R$. Suppose that $I_{\lambda^{-1}} \neq R$. Let q be a prime divisor of $I_{\lambda^{-1}}$. Then grade$(q) = 1$ because $I_{\lambda^{-1}}$ is a divisorial ideal of R. Since $\sqrt{I_\lambda} \subseteq q$ by (Step 2), we have $J_\lambda = I_\lambda + I_{\lambda^{-1}} \subseteq q$. Hence grade$(J_\lambda) = 1$. Thus we have $(0) \neq \mathcal{C}(\overline{R}/R) \subseteq q$ by Lemma 8.4.9. Since depth$(R_q) = 1$ and α is ultra-primitive over R_q, it follows that $I_{[\alpha]} \not\subseteq q$. Thus $R_q[\alpha]$ is a free R_q-module $R_q \oplus R_q\alpha \oplus \cdots \oplus R_q\alpha^{d-1}$ by Remark 8.4.2. Since $R_q[\alpha] = R_q[\lambda\alpha + \mu]$, we can choose $b_0, b_1, \ldots, b_{d-1} \in R_q$ such that $\lambda\alpha + \mu = b_0 + b_1\alpha + \cdots + b_{d-1}\alpha^{d-1}$. Since $1, \alpha, \ldots, \alpha^{d-1}$ are linearly independent over K in $K[\alpha]$, we have $\lambda = b_1 \in R_q$ and $\mu = b_0 \in R_q$. Hence $I_\lambda R_q = I_{b_1} R_q = R_q$, which means that $I_\lambda \not\subseteq q$, a contradiction, because $I_\lambda \subseteq \sqrt{I_\lambda} \subseteq q$ as was seen before. Hence $I_{\lambda^{-1}} = R$, that is, $\lambda^{-1} \in R$. Next we shall prove $\lambda \in R$. Put $a := \lambda^{-1} \in R$. Then $\lambda = 1/a$. Note that $R[\alpha + \mu/\lambda] = R[\alpha + a\mu] \subseteq R[\lambda\alpha + \mu] = R[\alpha]$, where the last equality follows from the assumption. Thus $\alpha + a\mu \in R[\alpha]$ yields that $a\mu \in R[\alpha] \cap K = R$. So we have $\mu = b/a$ for some $b \in R$. In this case, we have $\lambda = 1/a$ and $\mu = b/a$. Hence $R[\lambda\alpha + \mu] = R[\alpha/a + b/a] = R[(\alpha + b)/a]$. Put $\beta := \alpha + b$. Then $R[\beta] = R[(1/a)\beta]$. Noting that $b \in R$, it follows that $R[\alpha] = R[\beta]$, that β is exclusive over R because α is exclusive over R, and that β is ultra-primitive over R because $I_{[\alpha]} = I_{[\beta]}$. Considering β instead of α, we conclude that $\lambda = 1/a \in R$ by Proposition 8.4.7, which means that $\lambda \in U(R)$. In addition, $\mu \in R$ because $R = \sqrt{I_\lambda} = \sqrt{I_\mu}$. $\square$

8.5 Two Generators of Simple Extensions

Let R be a Noetherian domain, $R[X]$ be a polynomial ring, and α be an element of an algebraic extension field of the quotient field K of R. We investigate in this section the following equality: $R[f(\alpha), g(\alpha)] = R[\alpha]$, where $f(X), g(X) \in R[X]$. More precisely we consider the following equalities:

(1) $R[\alpha^2, \alpha^{2\ell+1}] = R[\alpha]$ for some/all $\ell > 0$;

(2) $R[\alpha^2] = R[\alpha]$;

(3) $R[f(\alpha), g(\alpha)] = R[\alpha]$, where $f(X)$ (respectively $g(X)$) $\in R[X]$ is a monic polynomial of degree 2 (respectively of odd degree $\ell \geq 3$).

We start with the following lemma.

Lemma 8.5.1 *Take $\beta, \gamma \in K$ and assume that the fractional ideal $(\beta, \gamma)R$ contains R. Then*

(i) $((\beta, \gamma)R)^{-1} = I_\beta \cap I_\gamma$, *where* $((\beta, \gamma)R)^{-1} := R :_K (\beta, \gamma)R$;

(ii) *if $(\beta, \gamma)R$ is invertible, then $((\beta, \gamma)R)^{-1} = (a, b)R$ and $a\beta + b\gamma = 1$ for some $a, b \in R$.*

Proof (i) Since $I := ((\beta, \gamma)R)^{-1} \subseteq R$, we have $I = ((\beta, \gamma)R)^{-1} \cap R = (R :_K (\beta, \gamma)R) \cap R = (R :_K \beta) \cap (R :_K \gamma) \cap R = (R :_R \beta) \cap (R :_R \gamma) = I_\beta \cap I_\gamma$.
(ii) Since $I(\beta, \gamma)R = R$, there exist $a, b \in I$ such that $a\beta + b\gamma = 1$. By the same reason as above, $\beta I \subseteq R$ and $\gamma I \subseteq R$. Take any $x \in I$. Then $x = ax\beta + bx\gamma \in (a, b)R$, whence $I \subseteq (a, b)R$. The reverse inclusion is trivial. ∎

Proposition 8.5.2 *Assume that $d \geq 2$. Then the following statements are equivalent:*

(*a*) $I_{[\alpha]} = I_{\eta_{d-1}} \cap I_{\eta_d}$ *and* $(\eta_{d-1}, \eta_d)R$ *is an invertible ideal such that* $R \subseteq (\eta_{d-1}, \eta_d)R$;

(*b*) *there exist* $a, b \in I_{[\alpha]}$ *such that* $a\eta_{d-1} + b\eta_d = 1$;

(*c*) $I_{[\alpha]} = (a, b)R$ *and* $a\eta_{d-1} + b\eta_d = 1$ *for some* $a, b \in R$;

(*d*) $I_{[\alpha]}(\eta_{d-1}, \eta_d)R = R$;

(*e*) $J_{[\alpha]} = R$ *and* $1, \eta_1, \cdots, \eta_{d-2} \in (\eta_{d-1}, \eta_d)R$.

Proof The implications: $(a) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b)$ are obvious.
$(b) \Rightarrow (a)$: Since $I_{[\alpha]}(\eta_{d-1}, \eta_d)R \subseteq R$, it follows that $1 = a\eta_{d-1} + b\eta_d \in I_{[\alpha]}(\eta_{d-1}, \eta_d)R$, and $R = I_{[\alpha]}(\eta_{d-1}, \eta_d)R$. Thus $(\eta_{d-1}, \eta_d)R = I_{[\alpha]}^{-1} \supseteq R$. Therefore $I_{[\alpha]} = ((\eta_{d-1}, \eta_d)R)^{-1} = I_{\eta_{d-1}} \cap I_{\eta_d}$ by Lemma 8.5.1.
$(a) \Rightarrow (e)$: Note first that $I_{[\alpha]} = I_{\eta_{d-1}} \cap I_{\eta_d} = ((\eta_{d-1}, \eta_d)R)^{-1}$ by Lemma 8.5.1. Since $(\eta_{d-1}, \eta_d)R$ is invertible, we have $((\eta_{d-1}, \eta_d)R)^{-1}(\eta_{d-1}, \eta_d)R = R$, and hence $((\eta_{d-1} = (a, b)R$ with $a\eta_{d-1} + b\eta_{d-1} = 1$ for some $a, b \in R$ by Lemma 8.5.1. In this case, we have $a, b \in ((\eta_{d-1}, \eta_d)R)^{-1} = I_{[\alpha]}$. So $I_{[\alpha]} = (a, b)R$. Therefore we conclude that $J_{[\alpha]} = R$ and $1, \ldots, \eta_{d-2} \in (\eta_{d-1}, \eta_d)R$.
$(e) \Rightarrow (a)$: Note that $R = J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)R = I_{[\alpha]}(\eta_{d-1}, \eta_d)R$. Hence $(\eta_{d-1}, \eta_d)R$ is invertible and $(\eta_{d-1}, \eta_d)R = I_{[\alpha]}^{-1} \supseteq R$. We have $I_{[\alpha]} = ((\eta_{d-1}, \eta_d)R)^{-1} = I_{\eta_{d-1}} \cap I_{\eta_d}$ by Lemma 8.5.1. ∎

Theorem 8.5.3 *Assume that α is an anti-integral element of degree $d \geq 2$ over R. Then the following statements are equivalent:*

(1) $R[\alpha^2, \alpha^{2\ell+1}] = R[\alpha]$ *for any positive integer* ℓ;

(2) $R[\alpha^2, \alpha^{2\ell+1}] = R[\alpha]$ *for some positive integer* ℓ;

(3) $I_{[\alpha]} = I_{\eta_{d-1}} \cap I_{\eta_d}$ *and* $(\eta_{d-1}, \eta_d)R$ *is an invertible ideal such that* $R \subseteq (\eta_{d-1}, \eta_d)R$*;*

(4) *there exist* $a, b \in I_{[\alpha]}$ *such that* $a\eta_{d-1} + b\eta_d = 1$*;*

(5) $I_{[\alpha]} = (a, b)R$ *and* $a\eta_{d-1} + b\eta_d = 1$ *for some* $a, b \in R$*;*

(6) $I_{[\alpha]}(\eta_{d-1}, \eta_d)R = R$*;*

(7) $R[\alpha]$ *is flat over* R *and* $1, \eta_1, \ldots, \eta_{d-2} \in (\eta_{d-1}, \eta_d)R$*;*

(8) $J_{[\alpha]} = R$ *and* $1, \eta_1, \ldots, \eta_{d-2} \in (\eta_{d-1}, \eta_d)R$.

Proof The equivalences: (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (8) follows from Proposition 8.5.2. Since α is anti-integral over R, $J_{[\alpha]} = R$ if and only if $R[\alpha]$ is flat over R by Theorem 2.3.6. So (7) $\Leftrightarrow$ (8) is valid.

(2) $\Rightarrow$ (4): Since $R[\alpha] = R[\alpha^2, \alpha^{2\ell+1}]$, we can write:

$$(\#) \quad \alpha = a_0 + a_{10}(\alpha^2) + a_{01}(\alpha^{2\ell+1}) + \sum_{i+j\geq 2} a_{ij}(\alpha^2)^i(\alpha^{2\ell+1})^j$$

for some $a_{ij} \in R$. Let

$$(*) \quad f(X) = a_0 - X + a_{10}(X^2) + a_{01}(X^{2\ell+1}) + \sum_{i+j\geq 2} a_{ij}(X^2)^i(X^{2\ell+1})^j$$

Then $f(\alpha) = 0$ by (#). Since α is anti-integral over R, it follows that $f(X) \in Ker(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. So we can write:

$$(**) \quad f(X) = \varphi_\alpha(X)(b_0 + b_1X + \cdots + b_mX^m)$$

for some $b_i \in I_{[\alpha]}$. Thus comparing $(*)$ and $(**)$, we have an equality: $b_1\eta_{d-1} + b_0\eta_d = -1$, and hence $a := -b_1, b := -b_0$ satisfy (4).
(5) $\Rightarrow$ (2): If $n \geq 2\ell + 1$ and n is odd, then $n = 2k + (2\ell + 1)$ for some nonnegative integer k, so that $\alpha^n \in R[\alpha^2, \alpha^{2\ell+1}]$. If $n \geq 2\ell + 1$ and n is even, then obviously $\alpha^n \in R[\alpha^2, \alpha^{2\ell+1}]$. Thus $\alpha^n \in R[\alpha^2, \alpha^{2\ell+1}]$ for all $n \geq 2\ell + 1$. By the assumption, there exist $b_0, b_1 \in I_{[\alpha]}$ such that $b_1\eta_{d-1} + b_0\eta_d = -1$. Fix $n \geq 2\ell + 1$ for a while. Now construct $b_0, b_1, b_2, \ldots, b_n \in I_{[\alpha]}$ as follows:
First let b_0, b_1 be the same as above. Let $i \geq 3$ be odd. Suppose that $\alpha^i \notin R[\alpha^2, \alpha^{2\ell+1}]$ and there exist $b_2, b_3, \ldots, b_i \in I_{[\alpha]}$ such that $b_0\eta_{d-i} + b_1\eta_{d-i+1} + \cdots + b_i\eta_d = 0$.
The case of $\alpha^{i+2} \notin R[\alpha^2, \alpha^{2\ell+1}]$: Put $-c = b_0\eta_{d-i} + b_1\eta_{d-i-1} + \cdots + b_i\eta_{d-2}$. Then $-c = (b_0\eta_{d-1} + b_1\eta_d)c$, and hence

$$b_0\eta_{d-i-2} + b_1\eta_{d-i-1} + \cdots + b_i\eta_{d-2} = -(cb_0)\eta_{d-1} - (cb_1)\eta_d$$

Thus putting $b_{i+1} := cb_0$ and $b_{i+2} := cb_1$, we have

$$b_0\eta_{d-i-2} + b_1\eta_{d-i-1} + \cdots + b_i\eta_{d-2} + b_{i+1}\eta_{d-1} + b_{i+2}\eta_d = 0$$

The case of $\alpha^{i+2} \in R[\alpha^2, \alpha^{2\ell+1}]$: Put $b_{i+1} := 0$ and $b_{i+2} := 0$.

Let $f(X) := \varphi_\alpha(X)(b_0 + b_1X + \cdots + b_nX^n)$. Then by construction of b_i's, we have $0 = f(\alpha) = b_0\eta_d + (b_0\eta_{d-1} + b_1\eta_d)\alpha + \gamma = b_0\eta_d - \alpha + \gamma$ with some $\gamma \in R[\alpha^2, \alpha^{2\ell+1}]$. Hence we conclude that $\alpha \in R[\alpha^2, \alpha^{2\ell+1}]$. ∎

Corollary 8.5.4 *Assume that α is an anti-integral element of degree $d = 2$ over R. Then $R[\alpha^2, \alpha^{2\ell+1}] = R[\alpha]$ for some/all $\ell \geq 1$ if and only if $R[\alpha]$ is flat over R.*

Corollary 8.5.5 *Assume that R is normal and that $d = 2$. Then $R[\alpha^2, \alpha^{2\ell+1}] = R[\alpha]$ for some/all $\ell \geq 1$ if and only if $R[\alpha]$ is flat over R.*

Corollary 8.5.6 *Assume that α is an anti-integral element of degree $d \geq 2$ over R. If $R[\alpha^2] = R[\alpha]$, then $I_{[\alpha]}(\eta_{d-1}, \eta_d)R = R$ and $R[\alpha]$ is flat over R.*

Proof Since $R[\alpha^2] \subseteq R[\alpha^2, \alpha^3] \subseteq R[\alpha] = R[\alpha^2]$, we have $R[\alpha^2, \alpha^3] = R[\alpha]$. Hence $I_{[\alpha]}(\eta_{d-1}, \eta_d)R = R$ and $R[\alpha]$ is flat over R by Theorem 8.5.3. ∎

Proposition 8.5.7 *Assume that $d \geq 2$. If $I_{[\alpha]}(\eta_{d-1}, \eta_d)R = R$, then $R[[\alpha^2]] = R[[\alpha]]$.*

Proof By the assumption, there exist $b_0, b_1 \in I_{[\alpha]}$ satisfying:

$$(*) \quad b_0\eta_{d-1} + b_1\eta_d = -1$$

Put $c_1 := b_0\eta_{d-3} + b_1\eta_{d-2} \in R$. Considering $(*) \times c_1$, we have $(c_1b_0)\eta_{d-1} + (c_1b_1)\eta_d = -c_1 = -(b_0\eta_{d-3} + b_1\eta_{d-2})$. Hence putting $b_2 := c_1b_0, b_3 := c_1b_1 \in I_{[\alpha]}$, we obtain

$$b_0\eta_{d-3} + b_1\eta_{d-2} + b_2\eta_{d-1} + b_3\eta_d = 0$$

In the sequel, consider η_i for $i \leq d$ and put $\eta_0 = 1$, $\eta_i = 0$ for $i < 0$. Next put $b_0\eta_{d-5} + b_1\eta_{d-4} + b_2\eta_{d-3} + b_3\eta_{d-2} =: -c_2 \in R$. Considering $(*) \times c_2$, we are in the similar situation. Put $b_4 := c_2b_0, b_5 := c_2b_1 \in I_{[\alpha]}$. Then we have the similar equality:

$$b_0\eta_{d-5} + b_1\eta_{d-4} + b_2\eta_{d-3} + b_3\eta_{d-2} + b_4\eta_{d-1} + b_5\eta_d = 0$$

Repeating this process, we obtain the sequence $b_0, b_1, b_2, b_3, \ldots$ in $I_{[\alpha]}$. Let $b(X) := b_0 + b_1X + b_2X^2 + b_3X^3 + \cdots \in R[[X]]$. Then $\varphi_\alpha(X)b(X) = a_0 - X + a_2X^2 + a_4X^4 + \cdots + a_{2n}X^{2n} + \cdots \in R[[X]]$. So $\alpha = a_0 + a_2(\alpha^2) + a_4(\alpha)^2 + \cdots + a_{2n}(\alpha^2)^n + \cdots \in R[[\alpha]]$. ∎

Proposition 8.5.8 *Assume that α is an anti-integral element of degree $d \geq 2$ over R and let $f(X)$ denote a monic polynomial in $X^2R[X]$ of odd degree ≥ 3. If $R[\alpha^2, f(\alpha)] = R[\alpha]$, then $I_{[\alpha]}(\eta_{d-1}, \eta_d) = R$.*

Proof It is obvious that $R[\alpha^2, \alpha^{2\ell+1}] \subseteq R[\alpha^2, \alpha^3]$ for all $\ell \geq 1$. So we have $R[\alpha^2, f(\alpha)] \subseteq \sum_{\ell\geq 1} R[\alpha^2, \alpha^{2\ell+1}] = R[\alpha^2, \alpha^3]$. Hence $R[\alpha^2, \alpha^3] = R[\alpha]$ since $R[\alpha^2, f(\alpha)] = R[\alpha]$. Thus we obtain $I_{[\alpha]}(\eta_{d-1}, \eta_d)R = R$ by Theorem 8.5.3. ∎

Proposition 8.5.9 *$R[\alpha]$ and $R[\alpha^2, f(\alpha)]$ are birational to each other for any monic polynomial $f(X)$ of odd degree $\ell \geq 1$.*

Proof If $\ell = 1$, then trivially $K(\alpha) = K(\alpha^2, f(\alpha))$. Next suppose that $\ell \geq 3$. We may assume that $f(\alpha) = u_1\alpha + u_3\alpha^3 + \cdots + u_\ell\alpha^\ell$ in $R[\alpha^2, f(\alpha)]$. Hence we have $f(\alpha) = \alpha(u_1 + u_3\alpha^2 + \cdots + u_\ell\alpha^{\ell-1}) \in K(\alpha^2, f(\alpha))$. Since $u_1 + u_3\alpha^2 + \cdots + u_\ell\alpha^{\ell-1} \in R[\alpha^2] \setminus (0)$, it follows that $\alpha \in K(\alpha^2, f(\alpha))$, and hence $K(\alpha) = K(\alpha^2, f(\alpha))$. ∎

Remark 8.5.10 Let $f(X) \in R[X]$ be a monic polynomial of odd degree $\ell \geq 3$ and $g(X) = X^2+tX+s \in R[X]$. Let B denote the subring $R[g(\alpha), f(\alpha)]$. Take a polynomial $h(X)$ of degree n. If n is even, then $n = 2k$ for some nonnegative integer k. Thus the polynomial $h(X) - ag(X)^k$ is of degree $\leq n-1$ for some $a \in R$. If $n \geq \ell$ and n is odd, then $n = \ell + 2k$ for some nonnegative integer k. So $h(X) - af(X)g(X)^k$ is of degree $\leq n-1$ for some $a \in R$. Hence any element $\beta \in R[\alpha]$ is expressed as $u_1\alpha + u_3\alpha^3 + \cdots + u_{\ell-2}\alpha^{\ell-2} + \gamma$, where $u_i \in R$ and $\gamma \in B$. Thus the R-module $R[\alpha]/R[g(\alpha), f(\alpha)]$ is finitely generated. ∎

Remark 8.5.11 Assume that (R, m) is a local domain and that $d \geq 2$, and let $f(X)$ denote a monic polynomial $X^\ell + u_1X^{\ell-1} + \cdots + u_{\ell-1}X + u_\ell$ in $R[X]$ of odd degree $\ell \geq 3$ such that $u_{2i} \in m$ $(1 \leq i \leq (\ell-1)/2)$. Then for any integer $n \geq \ell$, $\alpha^n \in R[\alpha^2, \alpha^\ell] + mR[\alpha] = R[\alpha^2, f(\alpha)] + mR[\alpha]$ and $R[\alpha^2, \alpha^\ell] + mR[\alpha] = R[g(\alpha), f(\alpha)] + mR[\alpha]$. ∎

Theorem 8.5.12 *Assume that (R, m) is a local domain and that α is an anti-integral element of degree $d \geq 2$, and let $f(X)$ denote a monic polynomial*

$X^\ell + u_1X^{\ell-1} + \cdots + u_{\ell-1}X + u_\ell$ in $R[X]$ of odd degree $\ell \geq 3$ such that $u_{2i} \in m$ $(1 \leq i \leq (\ell-1)/2)$. Then $R[\alpha^2, f(\alpha)] = R[\alpha]$ if and only if there exist $b_0, b_1 \in I_{[\alpha]}$ such that $b_0\eta_d + b_1\eta_{d-1} = 1$.

Proof Put $\ell = 2k+1$, $(k \geq 1)$. Then we have that $\alpha^\ell = f(\alpha) + c$, where $c \in R[\alpha^2, f(\alpha)] + mR[\alpha]$ because $\alpha^{2i} \in R[\alpha^2, f(\alpha)]$ and $u_{2i} \in m$ $(1 \leq i \leq (\ell-1)/2)$ (*cf.* Remark 8.5.1). Hence $R[\alpha^2, f(\alpha)] + mR[\alpha] = R[\alpha^2, \alpha^\ell] + mR[\alpha]$.
($\Rightarrow$): Assume that $R[\alpha] = R[\alpha^2, f(\alpha)]$. Then $R[\alpha] = R[\alpha^2, f(\alpha)] + mR[\alpha] = R[\alpha^2, \alpha^\ell] + mR[\alpha]$. Since the R-module $R[\alpha]/R[\alpha^2, \alpha^\ell]$ is finitely generated by Remark 8.5.11, we have $R[\alpha] = R[\alpha^2, \alpha^\ell]$ by NAK (*cf.* [M1]). So we conclude that $b_1\eta_{d-1} + b_0\eta_d = 1$ for some $b_0, b_1 \in I_{[\alpha]}$ by Theorem 8.5.2.
($\Leftarrow$): Assume that $b_1\eta_{d-1} + b_0\eta_d = 1$ for some $b_0, b_1 \in I_{[\alpha]}$. Then we have $R[\alpha] = R[\alpha^2, \alpha^\ell]$ by Theorem 8.5.2. So it follows that $R[\alpha] = R[\alpha^2, \alpha^\ell] + mR[\alpha] = R[\alpha^2, f(\alpha)] + mR[\alpha]$. Since the R-module $R[\alpha]/R[\alpha^2, f(\alpha)]$ is finitely generated over R by Remark 8.5.11, we have $R[\alpha] = R[\alpha^2, f(\alpha)]$ by NAK Lemma (*cf.* [M1]). ∎

Corollary 8.5.13 *Assume that (R, m) is a local domain and that α is an anti-integral element of degree $d \geq 2$. Let $g(X) = X^2 + tX + s \in R[X]$ with $t \in m$ and let $f(X)$ be a polynomial $X^\ell + u_1X^{\ell-1} + \cdots + u_{\ell-1}X + u_\ell \in R[X]$ of odd degree $\ell \geq 3$ with $u_2, u_4, \ldots, u_{\ell-1} \in m$. Then $R[g(\alpha), f(\alpha)] = R[\alpha]$ if and only if there exist $b_0, b_1 \in I_{[\alpha]}$ such that $b_1\eta_{d-1} + b_0d = 1$.*

Proof From Theorem 8.5.12, we have only to note that $R[g(\alpha), f(\alpha)] + mR[\alpha] = R[\alpha^2, \alpha^\ell] + mR[\alpha]$ and that the R-module $R[\alpha]/R[g(\alpha), f(\alpha)]$ is finitely generated, which follow from Remarks 8.5.10 and 8.5.11. ∎

Remark 8.5.14 Assume that R contains a field k of char $k \neq 2$. Let $g(X) = X^2 + tX + s$ with $s, t \in R$ and $f(X)$ be a monic polynomial in $R[X]$ of degree $\ell > 2$, where ℓ is odd. Then $g(X) = (X + \frac{1}{2}t)^2 + s - (\frac{1}{2}t)^2$. Put $\alpha^* := \alpha + \frac{1}{2}t \in R[\alpha]$. Then $R[\alpha^*] = R[\alpha]$ and $R[g(\alpha), f(\alpha)] = R[\alpha^{*2}, \alpha^{*\ell} + u_{\ell-2}\alpha^{*\ell-2} + \cdots + u_1\alpha^*]$ for some $u_j \in R$. Note here that if α is anti-integral over R then so is α^* by Proposition 3.1.13. ∎

Chapter 9

Flatness and Contractions of Ideals

Let R be a Noetherian domain with quotient field K. Let L be an algebraic field extension of K and $\alpha \in L$. Let A be an integral domain containing R. Assume that A and L are in some large fixed field. Let $K(A)$ denote the quotient field of A. Let $\pi^{(A)} : A[X] \longrightarrow A[\alpha]$ be the canonical A-algebra homomorphism sending X to α. Let $\varphi_\alpha^{(A)}(X)$ denote the minimal polynomial of α over $K(A)$ and $\deg \varphi_\alpha^{(A)}(X) = d_A$. Write

$$\varphi_\alpha^{(A)}(X) = X^{d_A} + \eta_1 X^{d_A - 1} + \cdots + \eta_{d_A}$$

where $\eta_i \in K(A)$. Let $I_{\eta_i}^A := A :_A \eta_i$ and $I_{[\alpha]}^A := \bigcap_{i=1}^{d_A} I_{\eta_i}^A$. It is easy to see that $I_{[\alpha]}^A = A[X] :_A \varphi_\alpha^{(A)}(X)$. If $A = R$, we put $d := d_A$, $I_{\eta_i} := I_{\eta_i}^A$, and $I_{[\alpha]} := I_{[\alpha]}^A$. It is easy to see that $I_{[\alpha]}$ (respectively $I_{[\alpha]}^A$) is an ideal of R (respectively A).

An element $\alpha \in L$ is called an *anti-integral element* over R of degree d if the equality $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$ holds. For a polynomial $f(X) \in R[X]$, $\mathsf{c}(f(X))$ denotes the ideal of R generated by the coefficients of $f(X)$. Let J be an ideal of $R[X]$; we denote by $\mathsf{c}(J)$ the ideal of R generated by the coefficients of the polynomials in J. If $\alpha \in L$ is anti-integral, it follows that $\mathsf{c}(\mathrm{Ker}(\pi)) = \mathsf{c}(I_{[\alpha]}\varphi_\alpha(X)R[X]) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. We put $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$, which is an ideal of R. It is not hard to see that $J_{[\alpha]} = \mathsf{c}(I_{[\alpha]}\varphi_\alpha(X))$. Similarly let $J_{[\alpha]}^A := I_{[\alpha]}^A(1, \eta_1, \ldots, \eta_{d_A})$. When α is an element in K, $\varphi_\alpha(X) = X - \alpha$. So we have $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = I_{[\alpha]} + \alpha I_{[\alpha]}$.

Throughout this chapter, we use the above notations unless otherwise specified.

9.1 Flatness of a Birational Extension

In Theorem 9.1.4, it will be shown that if $\beta_1, \ldots, \beta_n \in K$ are all strongly flat elements, then $R[\beta_1, \ldots, \beta_n]$ is flat over R. Example 9.1.5 shows that this assertion does not always hold if $\beta_i \notin K$ for some i, that is, the assertion above is not necessarily valid in the non-birational case. In this section, we treat this problem in the birational case.

Theorem 9.1.1 *Let $\{\alpha_i | 1 \leq i \leq n\}$ be a finite subset of K. Then $A := R[\alpha_1, \ldots, \alpha_n]$ is flat over R if and only if $I_{\alpha_1} \cdots I_{\alpha_n} A = A$, or equivalently $I_{\alpha_i} A = A$ for all i.*

Proof ($\Leftarrow$): Lemma 1.2.11 shows that A is flat over R if and only if $I_\alpha A = A$ for every $\alpha \in A$. This is applied to $\alpha = \alpha_i$ for each i.
($\Rightarrow$): Consider each $p \in \mathrm{Spec}(R)$. If $I_{\alpha_1} \cdots I_{\alpha_n} \not\subseteq p$, then I_{α_i} is not contained in p for all i, and hence $\alpha_i \in R_p$ for all i. Therefore it follows that $A_p = R_p$ in this case. Thus by Lemma 1.2.11, we see that A is flat over R. ∎

The following is an immediate consequence of Theorem 9.1.1.

Corollary 9.1.2 *Let $\{\alpha_i | 1 \leq i \leq n\}$ be a finite subset of K, and assume that $A := R[\alpha_1, \ldots, \alpha_n]$ is flat over R. Then we have*

$$\mathrm{Spec}(A) \cong \mathrm{Spec}(R) \setminus \bigcup_{i=1}^{n} V(I_{\alpha_i})$$

Corollary 9.1.3 *For $\alpha \in K$, $B := R[\alpha, \alpha^{-1}]$ is flat over R if and only if $I_\alpha B = B$.*

Proof If B is flat over R, then $I_\alpha I_{\alpha^{-1}} B = B$ by Theorem 9.1.1. Note that $I_{\alpha^{-1}} = \alpha I_\alpha$. Hence we have $\alpha (I_\alpha)^2 B = B$. Since α is a unit in B, we have $I_{\alpha^{-1}} B = \alpha I_\alpha B = B$. Thus B is flat over R by Theorem 9.1.1. ∎

Theorem 9.1.4 *Let $\{\alpha_i | 1 \leq i \leq n\}$ be a finite subset of K. If $R[\alpha_i]$ $(1 \leq i \leq n)$ are flat over R, then $R[\alpha_1, \ldots, \alpha_n]$ is flat over R.*

Proof Since α_i $(1 \leq i \leq n)$ are strongly flat elements over R, by definition, we have $R[\alpha_i]$ are flat over R, and hence $I_{\alpha_i} R[\alpha_i] = R[\alpha_i]$ by Theorem 9.1.1. Thus $I_{\alpha_i} R[\alpha_1, \ldots, \alpha_n] = R[\alpha_1, \ldots, \alpha_n]$ for all i $(1 \leq i \leq n)$. Therefore

$$I_{\alpha_1} \cdots I_{\alpha_n} R[\alpha_1, \ldots, \alpha_n] = R[\alpha_1, \ldots, \alpha_n]$$

This shows that $R[\alpha_1, \ldots, \alpha_n]$ is flat over R by Theorem 9.1.1. ∎

The following example shows that Theorem 9.1.4 does not hold in general for non-birational extensions.

Example 9.1.5 Let $R \subseteq A$ be a non-birational extension of Noetherian domains, and K and L be the quotient fields of R and A, respectively. Assume

that there are elements $\alpha \in L \setminus K$ and $\eta \in K$ such that $L = K[\alpha]$, $\alpha^2 = a \in R$, both η and $\eta^{-1} \notin R_p$ for some $p \in \mathrm{Spec}(R)$, and that $a\eta^2 \in R$. Hence we see that $R[\alpha]$ and $R[\beta]$ are free R-modules. Furthermore we can show that $A = R[\alpha, \beta]$ is not a flat R-module. Indeed, let us assume that $R[\alpha, \beta]$ is a flat R-module. By localizing at p, we may assume that R is a local ring with maximal ideal m. Note that $A = R + R\alpha + R\beta + R\alpha\beta$. Since A is a free R-module of rank 2, $A/mA = R/m + (R/m)\overline{\alpha} + (R/m)\overline{\beta} + (R/m)\overline{\alpha}\overline{\beta}$, $\overline{\alpha}, \overline{\beta} \in A/mA$ is a two-dimensional vector space over R/m. Since $\overline{1} \in A/mA$ is a linearly independent over R/m, we have the following possibilities:

(1) $A/mA = R/m + (R/m)\overline{\alpha}$,
(2) $A/mA = R/m + (R/m)\overline{\beta}$,
(3) $A/mA = R/m + (R/m)\overline{\alpha}\overline{\beta}$.

Therefore, by Nakayama's lemma, we have $A = R + R\alpha$, $A = R + R\beta$, or $A = R + R\alpha\beta$. If $\alpha \in R + R\alpha\beta$, then we have $\alpha = x + y\alpha\beta$ for some elements $x, y \in R$. Since η and $\eta^{-1} \notin R$ by assumption, we notice that x, y are nonzero elements of R. Now, it holds that $R \ni y^2\alpha^2\beta^2 = (\alpha - x)^2 = \alpha^2 - 2x\alpha + x^2$, and $\alpha = (a + x^2 - y^2\alpha^2\beta^2)/2x \in K$. This is a contradiction. Therefore $\alpha \notin R + R\alpha\beta$. In the same way, we see that $\beta \notin R + R\alpha$. Thus it follows that $\alpha \in R + R\alpha\beta$ or $\beta \in R + R\alpha$. Suppose that $\beta \notin R + R\alpha$. Then $\beta = x + y\alpha$ for some elements $x, y \in R$. Note that x, y are nonzero elements of R. Since $\beta^2 = x^2 + 2xy\alpha + \alpha^2 + a$, we have $2xy\alpha = \beta^2 - ax^2 \in R$. Since $xy \neq 0$, it holds that $\alpha \in K$, a contradiction. Therefore A is not a flat R-module. Finally, we construct the example satisfying these conditions. Let k be a field and X, Y, Z be the indeterminates over k. Let $R = k[X, Y]/(Y^2 - X^2Y + X^2) = k[x, y]$, where x, y denote the residue classes of X, Y in R. Let $A = R[Z]/(Z^2 - (y - 1)) = R[z]$, where z denotes the residue class of Z in A. Now, put $\alpha = z$, $\eta = y/x$, and $p = (x, y) \in \mathrm{Spec}(R)$. Then we see that $\alpha \notin K$. Now assume that $\eta \in R_p$. Then we have $y \in R_p$, and hence $pR_p = xR_p$. Thus R_p is a discrete valuation ring (DVR). This is a contradiction. Therefore we have $\eta \notin R_p$. Similarly, we see that $\eta^{-1} \notin R_p$. Further, we have $\eta^2 = y^2/x^2 = y - 1 = a \in R$ and $a\eta^2 = (y-1)^2 \in R$. Therefore it follows that $R[z, yz/x]$ is not a flat R-module but $R[z]$ and $R[yz/x]$ are flat R-modules by the preceding argument.

9.2 Flatness of a Non-Birational Extension

In this section, we treat the case that some of $\beta_1, \ldots, \beta_n$ are not contained in K and show that the similar assertion to Theorem 9.1.4 holds even in the non-birational case if $\beta_1, \ldots, \beta_n$ satisfy certain conditions.

It is known that if α is anti-integral and integral over R of degree d, then $R[\alpha]$ is a free R-module of rank d (*cf.* Corollary 2.3.5, Theorem 2.3.8).

We start with the following definition.

Definition 9.2.1 Assume that $\alpha \in L$ is anti-integral and integral over R of degree d. For an element $\beta \in K(\alpha) = K[\alpha]$, putting: $\beta = \lambda_0 + \lambda_1\alpha + \cdots + \lambda_{d-1}\alpha^{d-1}$ with $\lambda_i \in K$, let

$$T_\beta := \bigcap_{i=0}^{d-1} I_{\lambda_i} = R[\alpha] :_R \beta$$

Lemma 9.2.2 *Let A be a Noetherian domain containing R. Assume that $\alpha \in L$ is anti-integral over A of degree d_A. Put*

$$F_\ell^{(A)} := \{f(X) \in \operatorname{Ker}\pi^{(A)} | \deg f(X) = \ell\}$$

for a nonnegative integer ℓ, where $\pi^{(A)} : A[X] \longrightarrow A[\alpha]$ is the canonical A-algebra homomorphism sending X to α. If $\ell \geq d_A := [K(A)(\alpha) : K(A)]$, then $\mathsf{c}(F_\ell^{(A)}) = J_{[\alpha]}^{(A)}$.

Proof Since α is an anti-integral element of degree d_A over A, we have

$$\operatorname{Ker}(\pi^{(A)}) = I_{[\alpha]}^A \varphi_\alpha^{(A)}(X)A[X]$$

and hence

$$J_{[\alpha]}^A = \mathsf{c}(\operatorname{Ker}\pi^{(A)}) = \mathsf{c}(I_{[\alpha]}^A \varphi_\alpha^{(A)}(X)A[X])$$

Take $f(X) \in I_{[\alpha]}^A \varphi_\alpha^{(A)}(X)A[X]$ with $\deg f(X) = d_A$. For an integer $\ell \geq d_A$, we see that $X^{\ell-d_A} f(X) \in F_\ell^{(A)}$. Thus $\mathsf{c}(f(X)) = \mathsf{c}(X^{\ell-d_A} f(X)) \subseteq \mathsf{c}(F_\ell^{(A)})$. So we obtain that $J_{[\alpha]}^A \subseteq \mathsf{c}(F_\ell^{(A)})$. Conversely, take $f(X) \in F_\ell^{(A)}$. Then $f(X) \in \operatorname{Ker}\pi^{(A)} = I_{[\alpha]}^A \varphi_\alpha^{(A)}(X)A[X]$. Hence we can write:

$$f(X) = \sum f_i(X)g_i(X)$$

with $\deg f_i(X) = d_A$, $f_i(\alpha) = 0$. Hence, we have $\mathsf{c}(f(X)) \subseteq \sum \mathsf{c}(f_i(X))A \subseteq J_{[\alpha]}^A$. Thus $\mathsf{c}(F_\ell^{(A)}) \subseteq J_{[\alpha]}^A$. So we conclude that $J_{[\alpha]}^A = \mathsf{c}(F_\ell^{(A)})$. $\square$

Lemma 9.2.3 *Let $\beta \in L$ be an anti-integral element over R and let A be an Noetherian domain containing R. If β is an anti-integral element over A of degree e ($\leq d_A$), then $J_{[\beta]}A \subseteq J_{[\beta]}^A$.*

Proof Take $f(X) \in I_{[\beta]}\varphi_\beta(X)R[X]$ with $\deg f(X) = d_A$. Consider the natural exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\pi^{(A)}) \longrightarrow A[X] \xrightarrow{\pi^{(A)}} A[\beta] \longrightarrow 0$$

Then $f(X) \in \mathrm{Ker}\,\pi^{(A)}$. So by Lemma 9.2.2, we have $\mathrm{c}(f(X)) \subseteq \mathrm{c}(F_{d_A}^{(A)}) = J_{[\beta]}^A$, which yields that $J_{[\beta]}A \subseteq J_{[\beta]}^A$. ∎

Making these preparations, we have the following proposition.

Proposition 9.2.4 *Assume that $\alpha \in L$ is an anti-integral element over R of degree d. Let $\beta \in K[\alpha]$. Assume that β is anti-integral over both R and $R[\alpha]$. If $R[\beta]$ is flat over R, then $R[\alpha, \beta]$ is flat over $R[\alpha]$.*

Proof Put $A = R[\alpha]$. By Theorem 2.3.6, we have $J_{[\beta]} = R$. Hence $J_{[\beta]}^A = A$ by Lemma 9.2.3. Using Theorem 2.3.6 again, we have that $A[\beta]$ is flat over A. Hence $R[\alpha, \beta]$ is flat over $R[\alpha]$. ∎

Remark 9.2.5 In Theorem 2.4.5, it is shown that $\alpha \in L$ is anti-integral over R if and only if so is α^{-1}. ∎

Proposition 9.2.6 *Let L be an algebraic extension of a finite degree over K. Assume that $\alpha, \beta_1, \ldots, \beta_n \in L$, $[K(\alpha) : K] = d$, $L = K[\alpha]$, and that $R[\alpha]$ is flat over R. Consider $A = R[\alpha, \beta_1, \ldots, \beta_n]$. Then we can write $\beta_i = \eta_{i,0} + \eta_{i,1}\alpha + \cdots + \eta_{i,d-1}\alpha^{d-1}$ with $\eta_{i,j} \in K$ $(1 \le i \le n,\ 1 \le j \le d-1)$. If $T_{\beta_1} \cdots T_{\beta_n} A = A$, then A is flat over R.*

Proof Since $R[\alpha]$ is flat over R, we have only to show that A is flat over $R[\alpha]$. Let $P \in \mathrm{Spec}(R[\alpha])$ and $P \cap R = p$. If $p \supseteq T_{\beta_1} \cdots T_{\beta_n}$, then $pA = A$. This contradicts the fact that $P \supseteq pA$. Hence p does not contain $T_{\beta_1} \cdots T_{\beta_n}$, so that p does not contain T_{β_i} for all i. Thus by the definition of T_{β_i}, it holds that p does not contain $I_{\eta_{i,j}}$ for all i, j. Hence we see that $\beta_i \in R_p[\alpha]$. Therefore it follows that $A_p = R_p[\alpha]$. This implies that A_p is flat over R_p, and hence A is flat over R. ∎

Corollary 9.2.7 *Under the circumstances, if $T_{\beta_i} R[\alpha, \beta_i] = R[\alpha, \beta_i]$ for all i, we have $R[\alpha, \beta_1, \ldots, \beta_n]$ is flat over R.*

Proof Since $T_{\beta_1} \cdots T_{\beta_n} R[\alpha, \beta_1, \ldots, \beta_n] = R[\alpha, \beta_1, \ldots, \beta_n]$, the assertion is easily seen by Proposition 9.2.6. ∎

Now we show when $\beta \in L$ is flat over $R[\alpha]$ in terms of the ideal T_β. For this purpose, we study some properties of the ideal T_β.

Proposition 9.2.8 *Assume that $\alpha \in L$ is anti-integral and integral over R of degree d. Then the following statements hold:*

(i) *For $\beta_1, \ldots, \beta_n \in K[\alpha]$, put $A := R[\alpha, \beta_1, \ldots, \beta_n]$. If $(T_{\beta_1} \cap \ldots \cap T_{\beta_n})A = A$, then A is flat over R.*

(ii) *For $\beta \in K[\alpha]$, the equality $I_{[\beta]}^{R[\alpha]} \cap R = T_\beta$ holds.*

Proof (i) By the definitions of T_β and $I_{[\beta]}^{R[\alpha]}$, we have $T_\beta \subseteq I_{[\beta]}^{R[\alpha]}$. So the assumption yields $T_{\beta_i} A = A$ $(1 \leq i \leq n)$. Thus we have $I_{[\beta_i]}^{R[\alpha]} A = A$ $(1 \leq i \leq n)$. From this, it follows that $(I_{[\beta_1]}^{R[\alpha]} \cap \cdots \cap I_{[\beta_n]}^{R[\alpha]})A = A$, and hence A is flat over $R[\alpha]$. Since $R[\alpha]$ is flat over R, A is flat over R.
(ii) Note that $\beta \in K(\alpha)$. Since $I_{[\beta]}^{R[\alpha]} = R[\alpha] :_{R[\alpha]} \beta$ and $T_\beta = R[\alpha] :_R \beta$, we have $I_{[\beta]}^{R[\alpha]} \cap R = T_\beta$. ∎

Remark 9.2.9

(i) For $\beta \in K[\alpha]$, the rings $R[\alpha]$ and $R[\alpha][\beta]$ are birational. So it holds that $I_{[\beta^{-1}]}^{R[\alpha]} = \beta I_{[\beta]}^{R[\alpha]}$.

(ii) By use of (i) and Proposition 9.2.8 (ii), we have the following:

$$J_{[\beta]}^{R[\alpha]} = I_{[\beta]}^{R[\alpha]} + \beta I_{[\beta]}^{R[\alpha]} = I_{[\beta]}^{R[\alpha]} + I_{[\beta^{-1}]}^{R[\alpha]} \supseteq (T_\beta + T_{\beta^{-1}})R[\alpha]$$

and

$$J_{[\beta]}^{R[\alpha]} \cap R \supseteq T_\beta + T_{\beta^{-1}}$$

The next theorem is our main result in this section. ∎

Theorem 9.2.10 *Assume that α is anti-integral and integral over R of degree d. Let $\beta_1, \ldots, \beta_n$ be elements in $K[\alpha]$. If $J_{[\beta_i]} = R$ and each β_i is anti-integral over $R[\alpha]$ $(1 \leq i \leq n)$, then $R[\alpha, \beta_1, \ldots, \beta_n]$ is flat over R.*

Proof Since $\beta_1, \ldots, \beta_n \in K[\alpha]$, $A := R[\alpha, \beta_1, \ldots, \beta_n]$ is birational over $R[\alpha]$. By Theorem 2.3.6, each $R[\beta_i]$ $(1 \leq i \leq n)$ is flat over R. Thus A is flat over $R[\alpha]$. Since $R[\alpha]$ is flat over R by the assumption, A is flat over R by Theorem 9.1.4. ∎

Proposition 9.2.11 *Assume that α is anti-integral and integral over R. Let $\beta_1, \ldots, \beta_n \in K[\alpha]$. If $(T_{\beta_i} + T_{\beta_i^{-1}})R[\alpha] = R[\alpha]$ for each i, then $R[\alpha, \beta_1, \ldots, \beta_n]$ is flat over R.*

Proof By using Remark 9.2.9, we have $J_{[\beta_i]}^{R[\alpha]} = (T_{\beta_i} + T_{\beta_i^{-1}})R[\alpha] = R[\alpha]$ for each i because $R[\alpha]$ is faithfully flat over R. So $J_{[\beta_i]}^{R[\alpha]} = R[\alpha]$. Thus each β_i is a super flat (s-flat) element over $R[\alpha]$. Since $\beta_i \in K(\alpha)$, $R[\alpha, \beta_1, \ldots, \beta_n]$ is flat over $R[\alpha]$ by Theorem 9.1.4. Hence $R[\alpha, \beta_1, \ldots, \beta_n]$ is flat over R. ∎

In the preceding results, the condition that β is anti-integral over $R[\alpha]$ works effectively. We attempt to characterize this condition in terms of the ideal T_β.

Let $\mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) | \mathrm{depth}(R_p) = 1\}$. Let α be an element in an algebraic field extension L of K. We say that α is a *super-primitive element* of degree d if $J_{[\alpha]} \not\subseteq p$ for all $\in \mathrm{Dp}_1(R)$. Let A be a Noetherian domain containing R. An element $\beta \in K[\alpha]$ is a super-primitive element over A if and only if $\mathrm{grade}(J_{[\beta]}^A) > 1$. It is known in Theorem 2.2.8 that the super-primitive elements are anti-integral.

In the next proposition, we give a condition for α to be a super-primitive element over $R[\alpha]$.

Proposition 9.2.12 *Assume that α is anti-integral and integral over R of degree d. Let β be an element in $K[\alpha]$.*

(i) *β is super-primitive over $R[\alpha]$ if* $\mathrm{grade}(T_\beta + T_{\beta^{-1}}) > 1$. (*In particular, if* $\mathrm{grade}(T_\beta + T_{\beta^{-1}}) > 1$, *then β is anti-integral over $R[\alpha]$.*)

(ii) *β is a flat element over $R[\alpha]$ if $T_\beta + T_{\beta^{-1}} = R$.*

Proof (i) Since $R[\alpha]$ is faithfully flat and integral over R, $\mathrm{grade}(T_\beta + T_{\beta^{-1}}) > 1 \Rightarrow \mathrm{grade}(J_{[\beta]}^{R[\alpha]} \cap R) > 1 \Rightarrow \mathrm{grade}(J_{[\beta]}^{R[\alpha]}) > 1$. Our conclusion follows.
(ii) Since β is an s-flat element over $R[\alpha]$, we have $J_{[\beta]}^{R[\alpha]} = R[\alpha]$ and hence $J_{[\beta]}^{R[\alpha]} \cap R = R$. So we have $T_\beta + T_{\beta^{-1}} = R$ by the same way as in (i), and hence $J_{[\beta]}^{R[\alpha]} = R[\alpha]$ by Remark 9.2.9. So we conclude that β is a flat element over $R[\alpha]$ by Theorem 2.3.6. ∎

We close this section by investigating some relationships among the ideals $I_{[\beta]}$, $I_{[\beta]}^{R[\alpha]}$, and T_β.

Proposition 9.2.13 *Assume that α is anti-integral and integral over R of degree d. Let $\beta \in K[\alpha]$ which is anti-integral over both R and $R[\alpha]$. Then* $\sqrt{I_{[\beta]}} = \sqrt{T_\beta} = \sqrt{I_{[\beta]}^{R[\alpha]}} \cap R$.

Proof First we shall show the inclusion $I_{[\beta]} \subseteq I_{[\beta]}^{R[\alpha]} \cap R$. Let $\varphi_\beta(X) = X^d + \zeta_1 X^{d-1} + \cdots + \zeta_d$ with $\zeta_i \in K$. Then for a nonzero element $a \in I_{[\beta]}$, there

exists an algebraic dependence: $a\beta^d + a_1\beta^{d-1} + \cdots + a_d = 0,\ a_i = a\zeta_i \in R$. From this, we have $a_d(\beta^{-1})^d + \cdots + a_1\beta^{-1} + a = 0$. Put $f(X) := a_dX^d + \cdots + a_1X + a$. Then $f(\beta^{-1}) = 0$. By the assumption, β is anti-integral over both R and $R[\alpha]$. Hence β^{-1} is also anti-integral over both R and $R[\alpha]$ by Remark 9.2.5. So we have $f(X) = \sum f_i(X)g_i(X)$, where $\deg f_i(X) = 1$, $f_i(\beta^{-1}) = 0$ with $f_i(X), g_i(X) \in R[\alpha][X]$. Consider the constant term, we see that $a \in \beta^{-1}I_{[\beta^{-1}]}^{R[\alpha]} = \beta^{-1} \times \beta I_{[\beta]}^{R[\alpha]} = I_{[\beta]}^{R[\alpha]}$. Thus $I_{[\beta]} \subseteq I_{[\beta]}^{R[\alpha]} \cap R$. But since $I_{[\beta]}^{R[\alpha]} \cap R = T_\beta$ by Proposition 9.2.8, we have $I_{[\beta]} \subseteq T_\beta$. Thus $\sqrt{I_{[\beta]}} \subseteq \sqrt{T_\beta}$. Conversely, take $a \in T_\beta$. Then $a\beta \in R[\alpha]$. Since $R[\alpha]$ is a free R-module of rank d, $a\beta$ has a monic relation of degree d: $(a\beta)^d + b_1(a\beta)^{d-1} + \cdots + b_d = 0,\ b_i \in R$. From this, we have: $\beta^d + (b_1/a)\beta^{d-1} + \cdots + b_d/a^d = 0$. Thus $a^d \in I_{[\beta]}$. Therefore for a sufficiently large integer N, $T_\beta^N \subseteq I_{[\beta]}$ and hence $\sqrt{T_\beta} \subseteq \sqrt{I_{[\beta]}}$. The second equality follows from Proposition 9.2.8(ii). $\square$

Next we discuss certain relationships between flat extensions and unramified extensions. Let β be an anti-integral element in K. Then the universal module of differentials $\Omega_R(R[\beta])$ of $R[\beta]$ over R is given by

$$\Omega_R(R[\beta]) = R[\beta]/I_{[\beta]}\varphi'_\beta(\beta)R[\beta]$$

Since $\varphi_\beta(X) = X - \beta$ ($\beta \in K$), we have $I_{[\beta]}\varphi'_\beta(\beta) = I_{[\beta]}$. Thus

$$\begin{aligned} R[\beta] \text{ is unramified over } R &\iff \Omega_R(R[\beta]) = (0) \\ &\iff I_{[\beta]}R[\beta] = R[\beta] \\ &\iff R[\beta] \text{ is flat over } R \end{aligned}$$

(see Theorems 5.1.1, 5.1.4 for details). This means that, in the birational case, the flatness is equivalent to the unramifiedness. But in the non-birational case, there exists a counterexample to this assertion as follows:

Example 9.2.14 Let $\beta \in L$ satisfy $\beta^d = a \in R$ and $a \notin U(R)$ with $[K(\beta) : K] = d$ for $d > 1$. Then $\varphi_\beta(X) = X^d - a$ and $\varphi'_\beta(\beta) = d\beta^{d-1} \notin U(R[\beta])$. Thus $R[\beta]$ is integral and hence flat over R by Corollary 2.3.5, but not unramified over R.

Theorem 9.2.15 *Assume that α is anti-integral and integral over R of degree d. Let $\beta \in K[\alpha]$ and assume that β is anti-integral over both R and $R[\alpha]$.*

(i) *$R[\beta]$ is flat over R if and only if $R[\alpha, \beta]$ is unramified over $R[\alpha]$.*

(ii) *If $R[\beta]$ is unramified over R, then $R[\beta]$ is flat over R.*

Proof (i) By Theorem 9.2.10, $R[\beta]$ is flat over R if and only if $R[\alpha, \beta]$ is flat over $R[\alpha]$. Since $\beta \in K[\alpha]$, that is, $R[\alpha, \beta]$ and $R[\alpha]$ are birational, we have the equivalence: $R[\alpha, \beta]$ is flat over $R[\alpha]$ if and only if $R[\alpha, \beta]$ is unramified over $R[\alpha]$.
(ii) Assume that $R[\beta]$ is unramified over R. Then $\{f'(X) | f(X) \in I_{[\beta]}\varphi_\beta(X) R[X]\}$ generates the unit ideal in $R[\beta]$. Since $f(\beta) = 0$ and β is anti-integral over $R[\alpha]$, we have $f(X) \in I_{[\beta]}^{R[\alpha]}\varphi_\beta^{(R[\alpha])}(X)R[\alpha][X]$, where $\varphi_\beta^{(R[\alpha])}(X)$ denotes the monic minimal polynomial of β over $K(\alpha)$. Hence

$$f(X) = \sum F_i(X)G_i(X), \quad F_i(X) \in I_{[\beta]}^{R[\alpha]}\varphi_\beta^{(R[\alpha])}(X)R[\alpha][X]$$

where $G_i(X) \in R[\alpha][X]$, $\deg F_i(X) = 1$. Noting that $F_i'(\beta) \in I_{[\beta]}^{R[\alpha]}$, we have $f'(\beta) = \sum F_i'(\beta)G_i(\beta) \in I_{[\beta]}^{R[\alpha]}R[\alpha, \beta]$. Thus since $I_{[\beta]}^{R[\alpha]}R[\alpha, \beta] = R[\alpha, \beta]$, $R[\alpha, \beta]$ is flat over $R[\alpha]$. So by Proposition 9.2.4, we conclude that $R[\beta]$ is flat over R. ∎

Theorem 9.2.15 yields the following corollary.

Corollary 9.2.16 *Under the same assumptions as in Theorem 9.2.15, if $R[\beta]$ is unramified over R then $R[\alpha, \beta]$ is unramified over $R[\alpha]$.*

9.3 Anti-Integral Elements and Coefficients of its Minimal Polynomial

Let R be a Noetherian domain with quotient field K and L be an algebraic field extension of K. Let α be an element in L which is of degree d over K and $\varphi_\alpha(X) := X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ denote the minimal polynomial of α over K (that is, $\eta_i \in K$). Put $I_{\eta_i} := R :_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$, and put $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$ and $\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})$ as before.

The objective of this section is to investigate some relations between the extensions, $R[\alpha]$ over R and $R[\eta_1, \ldots, \eta_d]/R$.

Lemma 9.3.1 *(cf.* Theorem 2.3.6 and Proposition 4.4.1) *Assume that α is anti-integral over R. Then*

(1) *$R[\alpha]$ is flat over R if and only if $J_{[\alpha]} = R$;*
(2) *$R[\alpha]$ is faithfully flat over R if and only if $\tilde{J}_{[\alpha]} = R$.*

Lemma 9.3.2 *Assume that α is anti-integral over R. If $I_{[\alpha]}R[\alpha] = R[\alpha]$, then $R[\alpha]$ is flat over R.*

Proof Since $J_{[\alpha]} \supseteq I_{[\alpha]}$, the equality $I_{[\alpha]}R[\alpha] = R[\alpha]$ induces $\tilde{J}_{[\alpha]}R[\alpha] = R[\alpha]$. Hence $R[\alpha]$ is flat over R by Lemma 9.3.1. ∎

Proposition 9.3.3 *Assume that α is an anti-integral element over R. If $I_{[\alpha]}R[\alpha] = R[\alpha]$, then $R \hookrightarrow R[\eta_1, \ldots, \eta_d]$ is an open immersion.*

Proof Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, $R[\alpha]$ is flat over R by Lemma 9.3.3. So we have $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)R = R$ by Lemma 9.3.1. Take $p \in \mathrm{Spec}(R)$. If $I_{[\alpha]} \not\subseteq p$, then $\eta_i \in R_p$ for all i. Thus $R[\eta_1, \ldots, \eta_d]_p = R_p$. Assume that $I_{[\alpha]} \subseteq p$. Then $I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)R_p = R_p$ and hence $I_{[\alpha]}\eta_i R_p = R_p$ for some i. Thus we have $I_{[\alpha]}R_p = I_{\eta_i}R_p$. It follows that $I_{\eta_i}R[\eta_1, \ldots, \eta_d]_p = R[\eta_1, \ldots, \eta_d]_p$ and $pR[\eta_1, \ldots, \eta_d]_p = R[\eta_1, \ldots, \eta_d]_p$. Hence $R[\eta_1, \ldots, \eta_d]_p$ is flat over R_p by [Y, (1.1)]. Since $R[\eta_1, \ldots, \eta_d]$ and R are birational, $R \hookrightarrow R[\eta_1, \ldots, \eta_d]$ is an open immersion. ∎

Theorem 9.3.4 *Assume that α is anti-integral over R. If $R[\alpha]$ is flat over R, then $R[\eta_1, \ldots, \eta_d]$ and $R[\eta_1, \ldots, \eta_d][\alpha]$ are flat over R, and $R[\eta_1, \ldots, \eta_d][\alpha]$ is flat over $R[\eta_1, \ldots, \eta_d]$.*

Proof Since $R[\alpha]$ is flat over R, we have $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = R$. Since $\eta_1, \ldots, \eta_d \in R[\eta_1, \ldots, \eta_d]$, we have $I_{[\alpha]}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$. Take $p \in \mathrm{Spec}(R)$. If $p \not\supseteq I_{[\alpha]}$, we have $pR[\eta_1, \ldots, \eta_d] = R_p$ because $\eta_1, \ldots, \eta_d \in R_p$. If $p \supseteq I_{[\alpha]}$, we have $pR[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$. Hence $R[\eta_1, \ldots, \eta_d]$ is flat over R by [Y, (1.1)]. [Since $R[\eta_1, \ldots, \eta_d][\alpha]$ is a free $R[\eta_1, \ldots, \eta_d]$-module of rank d.] Thus $R[\eta_1, \ldots, \eta_d] \subseteq R[\eta_1, \ldots, \eta_d][\alpha]$ is a flat extension. Therefore $R \subseteq R[\eta_1, \ldots, \eta_d][\alpha]$ is also a flat extension. ∎

Example 9.3.5 Let R be a polynomial ring $k[a, b]$ over a field k. An element α is a root of an irreducible polynomial $\varphi_\alpha(X) = X^2 + (1/a)X + 1/b$. Then $\varphi_\alpha(X)$ is the minimal polynomial of α and α is an anti-integral element over R because R is a Noetherian normal domain. We have $J_{[\alpha]} = (a, b)R \neq R$ and $R[\eta_1, \ldots, \eta_d] = R[1/a, 1/b]$. Since $R[\eta_1, \ldots, \eta_d]$ is obtained by localizations, $R[\eta_1, \ldots, \eta_d]$ is flat over R. But since $J_{[\alpha]} \neq R$, $R[\alpha]$ is not flat over R. Thus the converse statement of Theorem 9.3.4 is not always valid.

Theorem 9.3.6 *The following statements are equivalent:*

(1) *$R[\alpha]$ is integral over R;*

(2) *$R[\eta_1, \ldots, \eta_d]$ is integral over R.*

Proof Let $\overline{R}$ denote the integral closure of R in K.
(2) $\Rightarrow$ (1): Since $R[\eta_1, \ldots, \eta_d]$ is integral over R, we have $R[\eta_1, \ldots, \eta_d] \subseteq \overline{R}$. Since α is integral over $R[\eta_1, \ldots, \eta_d]$, α is integral over $\overline{R}$. So α is integral over R.
(1) $\Rightarrow$ (2): Since R is a Noetherian domain, $\overline{R}$ is a Krull domain. So $\overline{R} = \bigcap \overline{R}_P$, $P \in \mathrm{Ht}_1(\overline{R})$, where $\overline{R}_P$ is a DVR. Since α is anti-integral and integral over a DVR $\overline{R}_P$, we have $\varphi_\alpha(X) \in \overline{R}_P[X]$. Hence $\eta_i \in \overline{R}_P$ for all i. So $\eta_i \in \overline{R}$, which implies that $R[\eta_1, \ldots, \eta_d]$ is integral over R. ∎

Lemma 9.3.7 *(cf.* The proof of Theorem 5.1.1). *Assume that α is an anti-integral element over R. Then $\Omega_R(R[\alpha]) \cong R[\alpha]/I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha]$.*

Theorem 9.3.8 *Assume that α is an anti-integral element over R. If $R[\alpha]$ is unramified over R, then $R[\eta_1, \ldots, \eta_d][\alpha]$ is unramified over $R[\eta_1, \ldots, \eta_d]$ and $R[\eta_1, \ldots, \eta_d]$ is unramified over R.*

Proof Note that $\Omega_R(R[\alpha]) \cong R[\alpha]/I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha]$ by Lemma 9.3.7. Since $R[\alpha]$ is unramified over R, we have $I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha] = R[\alpha]$. Thus $I_{[\alpha]}\varphi'_\alpha(\alpha)R[\alpha][\eta_1, \ldots, \eta_d] = R[\alpha][\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d][\alpha]$. Since $\varphi'_\alpha(\alpha) \in R[\alpha][\eta_1, \ldots, \eta_d]$, $\varphi'_\alpha(\alpha)$ is an invertible element in $R[\eta_1, \ldots, \eta_d][\alpha]$. Hence $R[\eta_1, \ldots, \eta_d][\alpha]$ is unramified over $R[\eta_1, \ldots, \eta_d]$. Note $R[\eta_1, \ldots, \eta_d][\alpha] = R[\eta_1, \ldots, \eta_d][X]/\varphi_\alpha(X)R[\eta_1, \ldots, \eta_d][X]$. So $R[\eta_1, \ldots, \eta_d][\alpha]$ is flat over $R[\eta_1, \ldots, \eta_d]$. Moreover $R[\eta_1, \ldots, \eta_d][\alpha] = I_{[\alpha]}\varphi'_\alpha(\alpha)R[\eta_1, \ldots, \eta_d][\alpha] \subseteq I_{[\alpha]}R[\eta_1, \ldots, \eta_d][\alpha] \subseteq R[\eta_1, \ldots, \eta_d][\alpha]$. Hence $I_{[\alpha]}R[\eta_1, \ldots, \eta_d][\alpha] = R[\eta_1, \ldots, \eta_d][\alpha]$. Since $R[\eta_1, \ldots, \eta_d][\alpha]$ is flat over $R[\eta_1, \ldots, \eta_d]$ and $R[\eta_1, \ldots, \eta_d][\alpha]$ is integral over $R[\eta_1, \ldots, \eta_d]$, $R[\eta_1, \ldots, \eta_d][\alpha]$ is faithfully flat over $R[\eta_1, \ldots, \eta_d]$. So $I_{[\alpha]}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$. Thus $R \hookrightarrow R[\eta_1, \ldots, \eta_d]$ is an open immersion by Proposition 9.3.3 and hence unramified. ∎

Now we characterize the ring $R[\alpha] \cap K$ under the condition $I_{[\alpha]}R[\alpha] = R[\alpha]$.

Lemma 9.3.9 *Let p be a prime ideal of R. If $pR[\alpha] = R[\alpha]$, then α^{-1} is integral over R_p.*

Proof Since $pR[\alpha] = R[\alpha]$, we have $a_0 + a_1\alpha + \cdots + a_\ell\alpha^\ell = 1$ for some $a_i \in p$. Thus α satisfies the equation: $a_\ell\alpha^\ell + \cdots + a_1\alpha + (a_0 - 1) = 0$. Hence α^{-1} satisfies the equation: $a_\ell + a_1\alpha^{-1} + \cdots + a_1\alpha^{\ell-1} + (a_0 - 1)(\alpha^{-1})^\ell = 0$. Since $a_0 - 1$ is a unit in R_p. Thus we can conclude that α^{-1} is integral over R_p. ∎

Proposition 9.3.10 *Assume that α is an anti-integral element over R. Consider the following statements:*

(1) $I_{[\alpha]}R[\alpha] = R[\alpha]$;
(2) $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$;
(3) $I_{[\alpha]} = I_{\eta_d}$ *and* $I_{\eta_d}(1, \eta_d)R = R$.

Then the following implications hold: (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3).

Proof (1) $\Rightarrow$ (2): There exists $p \in \mathrm{Spec}(R)$ such that $I_{[\alpha]} + I_{[\alpha^{-1}]} \subseteq p$. Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, α^{-1} is integral over R_p by Lemma 9.3.9. Since α is anti-integral over R, so is α^{-1} by Theorem 2.4.5. Hence α^{-1} is anti-integral and integral over R_p. Thus $\varphi_{\alpha^{-1}}(X) \in R_p[X]$ and hence $I_{[\alpha^{-1}]}R_p = R_p$, which contradicts the assumption $I_{[\alpha^{-1}]} \subseteq p$.
(2) $\Leftrightarrow$ (3): Since $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$ by Lemma 2.4.6. we have $I_{[\alpha]} + I_{[\alpha^{-1}]} = I_{[\alpha]}(1, \eta_d)R = R$. So we have $I_{[\alpha]} = I_{\eta_d}$ and $J_{\eta_d} = I_{\eta_d}(1, \eta_d)R = R$. The converse implication can be seen by tracing the above argument backward. $\square$

Example 9.3.11 The following example shows that the implication (2) $\Rightarrow$ (1) is not valid in general. Let R be a polynomial ring $k[a, b]$ over a field k. Let α be a solution of the equation: $\varphi_\alpha(X) := X^2 + (b/a^2)X + ((a-1)/a)^2 = 0$. Then α is anti-integral over R because R is a Noetherian normal domain. We have $I_{[\alpha]} = a^2R$, $\varphi_{\alpha^{-1}}(X) = X^2 + (b/(a-1)^2)X + (a/(a-1))^2$, and $I_{[\alpha^{-1}]} = (a-1)^2R$. Thus $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$. Moreover we have $J_{[\alpha]} = R$ and $\tilde{J}_{[\alpha]} = a^2(1, b/a^2)R = (a^2, b)R$. Since $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$, we have $\sqrt{\tilde{J}_{[\alpha]}} \neq \sqrt{I_{[\alpha]}}$. Hence $I_{[\alpha]}R[\alpha] \neq R[\alpha]$, which implies that the implication (2) $\Rightarrow$ (1) does not always hold.

Recall that an element $\alpha \in L$ is called *exclusive* over R if $R[\alpha] \cap K = R$. Now we study the exclusiveness for a while. We start with recalling the following lemma.

Lemma 9.3.12 (Theorem 7.2.7) *Assume that R contains an infinite field k and that α is a super-primitive element over R. Then the following statements are equivalent:*

(1) *α is exclusive over R;*
(2) $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$;
(3) $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ *or* $\tilde{J}_{[\alpha]} = R$.

Proposition 9.3.13 *Assume that α is super-primitive over R and that R contains an infinite field. If either* $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ *or* $\tilde{J}_{[\alpha]} = R$, *then both α and α^{-1} are exclusive, i.e., $R[\alpha] \cap K = R[\alpha^{-1}] \cap K = R$.*

Proof By Lemma 9.3.12, we have the following equivalences:
(a) α is exclusive over $R \Leftrightarrow \mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$;
(b) α^{-1} is exclusive over $R \Leftrightarrow \mathrm{grade}(I_{[\alpha^{-1}]}(\eta_1/\eta_d, \ldots, \eta_{d-1}/\eta_d, 1)) > 1$
$\Leftrightarrow \mathrm{grade}(I_{[\alpha]}(\eta_1, \ldots, \eta_d)) > 1$;
where the last equivalence follows from Lemma 2.4.6. These equivalence induce our conclusion. ∎

Proposition 9.3.14 *Assume that α is a super-primitive element over R. If $R[\alpha]$ is faithfully flat over R, then α is exclusive.*

Proof From Lemma 9.3.1, it follows the equivalence: $R[\alpha]$ is faithfully flat over $R \Leftrightarrow \tilde{J}_{[\alpha]} = R$. So we have our conclusion by Lemma 9.3.12. ∎

Proposition 9.3.15 *Assume that α is super-primitive over R and that R contains an infinite field. If $R[\alpha]_p$ is faithfully flat over R_p for each $p \in \mathrm{Dp}_1(R)$, then α is exclusive, i.e., $R[\alpha] \cap K = R$.*

Proof By Lemma 9.3.12, note that $R[\alpha]_p$ is faithfully flat over R_p for each $p \in \mathrm{Dp}_1(R) \Leftrightarrow \mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$ or $\tilde{J}_{[\alpha]} = R$ by Lemma 9.3.1. The latter condition gives rise to the statement that α is exclusive over R by Lemma 9.3.12. ∎

Lemma 9.3.16 *Assume that α is super-primitive over R. If $I_{[\alpha]}R[\alpha] = R[\alpha]$, then $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$.*

Proof Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, $R[\alpha]$ is flat over R by Lemma 9.3.2. Take $P \in \mathrm{Dp}_1(R[\alpha])$ and put $p := P \cap R$. Then $p \in \mathrm{Dp}_1(R)$. Since α is super-primitive over R, the ideal $I_{[\alpha]}R_p$ is a principal ideal. So there exists $a \in I_{[\alpha]}$ such that $I_{[\alpha]}R_p = aR_p$. Hence $aR[\alpha]_p = R[\alpha]_p$ by the assumption $I_{[\alpha]}R[\alpha] = R[\alpha]$. [Since $I_{[\alpha]} \subseteq I_{\eta_i}$ by definition, putting $\eta_i = b_i/a$ with $b_i \in R$.] Since a is an invertible element in $R[\alpha]_p$, we have $\eta_i \in R[\alpha]_p \subseteq R[\alpha]_P$. Thus $\eta_i \in \bigcap_{P \in \mathrm{Dp}_1(R[\alpha])} R[\alpha]_P = R[\alpha]$. Therefore $R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$. ∎

Now we give an equivalent condition to $I_{[\alpha]}R[\alpha] = R[\alpha]$.

Theorem 9.3.17 *Assume that α is a super-primitive element over R. The following statements are equivalent:*

(1) $I_{[\alpha]}R[\alpha] = R[\alpha]$;

(2) $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$ *and* $I_{[\alpha]}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$.

If the condition (2) *holds,* $R[\eta_1, \ldots, \eta_d]$ *is flat over* R.

Proof (1) $\Rightarrow$ (2): The first statement is shown in Lemma 9.3.16. The assumption $I_{[\alpha]}R[\alpha] = R[\alpha]$ implies that $R[\alpha]$ is flat over R by Theorem 9.3.4 and $R[\eta_1, \ldots, \eta_d]$ is flat over R by Lemma 9.3.2. Hence $J_{[\alpha]} = R$ by Proposition 2.3.6. Since α is anti-integral over $R[\eta_1, \ldots, \eta_d]$ (*cf.* Theorem 7.1.15 and since α is integral over $R[\eta_1, \ldots, \eta_d]$, it follows that $I_{[\alpha]}^{R[\eta_1,\ldots,\eta_d]} = R[\eta_1, \ldots, \eta_d]$, where $I_{[\alpha]}^{R[\eta_1,\ldots,\eta_d]} = R[\eta_1, \ldots, \eta_d] :_{R[\eta_1,\ldots,\eta_d]} \varphi_\alpha(X)$. Thus $I_{[\alpha]}R[\eta_1, \ldots, \eta_d] = I_{[\alpha]}^{R[\eta_1,\ldots,\eta_d]}$ because $R[\eta_1, \ldots, \eta_d]$ is flat over R. Moreover we have $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$ by Lemma 9.3.16.
(2) $\Rightarrow$ (1): Since $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$ and $I_{[\alpha]}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$, we have $I_{[\alpha]}R[\alpha] = R[\alpha]$. $\square$

Proposition 9.3.18 *Assume that α is a super-primitive element over R and that R contains an infinite field. If $R[\eta_d]$ is flat over R, then $R[\alpha] \cap K \subseteq R[\eta_d]$.*

Proof Since R and $R[\eta_d]$ have the same quotient field K, the element α is of degree d over both R and $R[\eta_d]$. Put $I_{[\alpha]}^{R[\eta_d]} := \bigcap_{i=1}^{d} I_{\eta_i}^{R[\eta_d]}$, where $I_{\eta_i}^{R[\eta_d]} := R[\eta_d] :_{R[\eta_d]} \eta_i$. Then $I_{[\alpha]} \subseteq I_{[\alpha]}^{R[\eta_d]}$, so that $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) \subseteq J_{[\alpha]}^{R[\eta_d]} = I_{[\alpha]}^{R[\eta_d]}(1, \eta_1, \ldots, \eta_d)$, where $J_{[\alpha]}^{R[\eta_d]} := [\alpha]^{R[\eta_d]}(1, \eta_1, \ldots, \eta_d)$. Since α is super-primitive over R, we have $\mathrm{grade}(J_{[\alpha]}) > 1$. Since $R[\eta_d]$ is flat over R, we have $\mathrm{grade}(J_{[\alpha]}R[\eta_d]) > 1$ and hence $\mathrm{grade}(J_{[\alpha]}^{R[\eta_d]}) > 1$. So α is super-primitive over $R[\eta_d]$. Since $\eta_d \in R[\eta_d]$, we have $\tilde{I}_{[\alpha]}^{R[\eta_d]} \subseteq I_{\eta_d}^{R[\eta_d]}$. So applying Lemma 9.3.12 to the extension $R[\alpha]$ of $R[\eta_d]$, we obtain $R[\alpha] \cap K \subseteq R[\eta_d][\alpha] \cap K = R[\eta_d]$. $\square$

Theorem 9.3.19 *Assume that α is super-primitive over both R and $R[\eta_d]$ and that R contains an infinite field. Consider the following statements:*

(1) $I_{[\alpha]}R[\alpha] = R[\alpha]$;

(2) $R[\eta_d] \subseteq R[\alpha]$, $I_{[\alpha]} = I_{\eta_d}$ *and* $R[\eta_d]$ *is flat over* R;

(3) $R[\alpha] \cap K = R[\eta_d] = R[\eta_1, \ldots, \eta_d]$.

Then the implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) *hold.*

Proof (1) + (2) $\Rightarrow$ (3): (1) implies that $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$ by Lemma 9.3.16. Since $R[\eta_d]$ is flat over R, $R[\eta_d] \supseteq R[\alpha] \cap K$ by Proposition 9.3.18. Hence we have $R[\alpha] \cap K = R[\eta_d] \supseteq R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_{d-1}]$.
(1) $\Rightarrow$ (2): We have $R[\eta_d] \subseteq R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$ by Lemma 9.3.16, and $I_{[\alpha]} = I_{\eta_d}$ by Proposition 9.3.10. Since $R \hookrightarrow R[\eta_d] \hookrightarrow R[\eta_1, \ldots, \eta_d]$ is an open immersion by Proposition 9.3.3, $R \hookrightarrow R[\eta_d]$ is flat.
(2) $\Rightarrow$ (1): Since $R[\eta_d]$ is flat over R, we have $I_{\eta_d}^{R[\eta_d]} = I_{\eta_d} R[\eta_d]$. Thus the fact $\eta_d \in R[\eta_d]$ implies that $I_{\eta_d}^{R[\eta_d]} = R[\eta_d]$. So it follows that $I_{\eta_d} R[\alpha] = R[\alpha]$ because $R[\eta_d] \subseteq R[\alpha]$. Since $I_{[\alpha]} = I_{\eta_d}$, we conclude $I_{[\alpha]} R[\alpha] = R[\alpha]$. ∎

Remark 9.3.20 Assume that α is anti-integral over R and that $\eta_d \in R$. Then $R[\alpha]$ is faithfully flat over R if and only if $R[\alpha]$ is flat over R. Indeed, since $\eta_d \in R$, we have $I_{[\alpha]} = \bigcap_{i=1}^{d} I_{\eta_i} = \tilde{I}_{[\alpha]}$ and hence $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}) = \tilde{J}_{[\alpha]}$. Hence $\tilde{J}_{[\alpha]} = R$ if and only if $J_{[\alpha]} = R$. Thus our conclusion follows from Lemma 9.3.1. ∎

Proposition 9.3.21 *Assume that α is a super-primitive element of degree d over R. Assume that the polynomial $\psi(X) := X^{d-1} + \eta_1 X^{d-2} + \cdots + \eta_{d-1}$ is irreducible in $K[X]$ and let β be a solution of $\psi(X) = 0$. Further assume that $\eta_d \in R$. Then β is anti-integral over R, and $R[\alpha]$ is flat over R if and only if $R[\beta]$ is flat over R.*

Proof Since $\eta_d \in R$, noting that $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$ by definition, we conclude that $I_{[\alpha]} = \bigcap_{i=1}^{d} I_{\eta_i} = \bigcap_{i=1}^{d-1} = I_{[\beta]}$ and hence $J_{[\beta]} = I_{[\beta]}(1, \eta_1, \ldots, \eta_{d-1}) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}) = J_{[\alpha]}$. ∎

Theorem 9.3.22 *Assume that K contains a field of characteristic zero and that $\eta_d \in R$. Let β be a solution of $\varphi'_\alpha(X) = 0$, where $\varphi'_\alpha(X) = dX^{d-1} + (d-1)\eta_1 X^{d-2} + \cdots + \eta_{d-1}$. Then*

(1) *if α is super-primitive over R, so is β;*
(2) *$R[\alpha]$ is flat over R if and only if $R[\beta]$ is flat over R.*

Proof By the similar argument in the proof of Proposition 9.3.21, we have $J_{[\alpha]} = J_{[\beta]}$. ∎

Example 9.3.23 Consider the case $d = 2$. Put $\varphi_\alpha(X) := X^2 + \eta X + a$ with $a \in R$. Let α be a solution of an equation $\varphi_\alpha(X) = 0$. Then $R[\alpha]$ is flat over

R if and only if $R[\eta]$ is flat over R because α is super-primitive over R. In this case, α is characterized by η.

Lemma 9.3.24 *If $I_{[\alpha]}$ is an invertible ideal of R, then α is a super-primitive element over R.*

Proof For each $p \in \mathrm{Spec}(R)$, $(I_{[\alpha]})_p$ is a principal ideal of R_p. So the conclusion follows Theorem 2.3.10. ∎

We say that an element $\alpha \in L$ is an *s-flat* element over R if $J_{[\alpha]} = R$.

Proposition 9.3.25

(1) *An element $\alpha \in L$ is an s-flat element over R if and only if so is α^{-1}.*

(2) *$\alpha \in L$ is s-flat over R if and only if α is anti-integral/super-primitive over R and $R[\alpha]$ is flat over R.*

Proof (1) Since $J_{[\alpha]} = J_{[\alpha^{-1}]}$ by Proposition 2.4.7, we conclude that $J_{[\alpha]} = R$ if and only if $J_{[\alpha^{-1}]} = R$.
(2) ($\Rightarrow$): $J_{[\alpha]} = R$ by definition, and hence $J_{[\alpha]} \not\subseteq p$ for every $p \in \mathrm{Dp}_1(R)$, which means that α is super-primitive over R. A super-primitive element is an anti-integral element by Theorem 2.2.8. In this case, $J_{[\alpha]} = R$ implies that $R[\alpha]$ is flat over R by Theorem 2.2.6.
($\Leftarrow$): Assume that α is anti-integral over R and $R[\alpha]$ is flat over R. Then we have $J_{[\alpha]} = R$ by Theorem 2.3.6. Thus α is an s-flat element over R. ∎

Proposition 9.3.26 *When $\alpha \in K$ is s-flat over R if and only if both $R[\alpha]$ and $R[\alpha^{-1}]$ are flat over R.*

Proof This follows from Proposition 1.2.12 and Theorem 2.3.6. ∎

Proposition 9.3.27 *For some i $(1 \le i \le d)$, assume that $I_{[\alpha]} = I_{\eta_i}$ and that η_i is an s-flat element over R. Then α is an s-flat element over R. Moreover if $i \neq d$, then $R[\alpha]$ is faithfully flat over R.*

Proof Since η_i is an s-flat element over R, we have $J_{\eta_i} = I_{\eta_i}(1, \eta_i) = R$ by definition. Since $I_{[\alpha]} = I_{\eta_i}$ and $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) \supseteq J_{\eta_i} = R$, we have $J_{[\alpha]} = R$. So α is an s-flat element over R. Assume that $i \neq d$.

Then $\tilde{J}_{[\alpha]} \supseteq I_{\eta_i}(1, \eta_i) = R$ and hence $\tilde{J}_{[\alpha]} = R$. Thus our conclusion follows from Lemma 9.3.1. $\square$

Theorem 9.3.28 *Assume that (R, m) is a local ring. Then the following statements are equivalent:*

(i) *α is an s-flat element over R;*

(ii) *for some i $(1 \leq i \leq d)$, the following conditions hold: η_i is an s-flat element over R and $I_{[\alpha]} = I_{\eta_i}$.*

Proof (ii) $\Rightarrow$ (i) is shown in Proposition 9.3.27.
(i) $\Rightarrow$ (ii): We have only to show this in the case $I_{[\alpha]} \subseteq m$. We have $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = R$ because α is an s-flat element over R. So there exists i such that $\eta_i I_{[\alpha]} = R$ because R is local. Thus $I_{[\alpha]} = I_{\eta_i}$ and $J_{\eta_i} = I_{\eta_i}(1, \eta_i) \supseteq I_{\eta_i}\eta_i = I_{[\alpha]}\eta_i = R$, which yields that η_i is an s-flat element over R. $\square$

Remark 9.3.29 Let (R, m) be a local ring. If there exists a prime ideal p of R such that none of $\eta_1, \ldots, \eta_d$ is a strongly flat element over R_p, then $R[\alpha]$ is not flat over R. Such p is the one not containing $J_{[\alpha]}$. $\square$

Example 9.3.30 Let R be a local ring $k[a, b]_{(a,b)}$, where $k[a, b]$ is a polynomial ring over a field k.

(1) Let α be a solution of the equation: $\varphi_\alpha(X) := X^2 + (b/a)X + a/b = 0$. Then $\varphi_\alpha(X)$ is a minimal polynomial of α and α is super-primitive over R because R is a Noetherian normal domain. We have $I_{[\alpha]} = abR$ and $J_{[\alpha]} = I_{[\alpha]}(1, b/a, a/b)R = ab(b/a, a/b)R = (b^2, a^2)R \neq R$. So $R[\alpha]$ is not flat over R. We see that $\eta_1 := b/a$ and $\eta_2 := a/b$ and that neither I_{η_1} nor I_{η_2} is equal to R. Note here that $I_{[\alpha]} \neq I_{\eta_1}$ and $I_{[\alpha]} \neq I_{\eta_2}$.

(2) Let α be a solution of the equation: $\varphi_\alpha(X) := X^3 + (b/a)X^2 + (a/b)X + 1/a = 0$. Then α is super-primitive over R as in (1). It follows that $1/a$ is an s-flat element. But $I_{[\alpha]} = abR$ is equal to none of $I_{b/a}$, $I_{a/b}$, and $I_{1/a}$. Since $J_{[\alpha]} \neq R$, $R[\alpha]$ is not flat over R.

Theorem 9.3.31 *Assume that $I_{[\alpha]}$ is an invertible ideal of R. If $R[\alpha]$ is flat over R, then for each $p \in \mathrm{Spec}(R)$, there exists i such that each η_i is an s-flat element over R_p and that $I_{[\alpha]}R_p = I_{\eta_i}R_p$.*

Proof Since $I_{[\alpha]}$ is an invertible ideal, α is super-primitive over R by Lemma 9.3.24. So $R[\alpha]$ is flat over R if and only if $J_{[\alpha]} = R$. Take $p \in \mathrm{Spec}(R)$. Localizing at p, we may assume that R is a local ring with maximal ideal m.

Since $I_{[\alpha]}$ is invertible, we have $I_{[\alpha]} = aR$ and $\eta_i = b_i/a$ for some $a, b_i \in R$. Assume first that $a \notin m$. Then $\eta_i \in R$ and hence η_i is an s-flat element over R for all i $(1 \leq i \leq d)$. Assume next that $a \in m$. Then $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = (a, b_1, \ldots, b_d) = R$ and hence there exists i such that $b_i \notin m$. So $\eta_i = b_i/a$ is an s-flat element and $I_{[\alpha]} = aR = I_{\eta_i}$. ∎

9.4 Denominator Ideals and Flatness (II)

Let α be an anti-integral element of degree d over R. Consider the following statements:

(i) $I_{[\alpha]}R[\alpha] = R[\alpha]$;

(ii) $R[\alpha]$ is flat over R;

(iii) $R[\alpha]$ is unramified over R.

We have shown that the statements above are all equivalent if $d = 1$, i.e., $\alpha \in K$ (*cf.* Theorem 5.2.8 It seems natural to ask whether the above equivalences are valid in the case $d > 1$. Our objective is to investigate the implications among (i), (ii), and (iii).

We begin with the following known result.

Proposition 9.4.1 *Let α be an anti-integral element of degree d over R. Then the implications* : (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (ii), *and* (iii) $\Rightarrow$ (i) *hold.*

Proof The implication (i) $\Rightarrow$ (ii) is seen in Theorem 3.4.1. The implications (iii) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are seen in Theorem 5.1.4 and Remark 5.1.3, respectively. ∎

The following examples and the remark show that the reverse implications in Proposition 9.4.1 are not always valid.

Example 9.4.2 We construct an example which shows that the implication (ii) $\Rightarrow$ (i) is not always valid. Let k be a field and let $R := k[x, y]$ be a polynomial ring. Let α be an algebraic element whose minimal polynomial is $\varphi_\alpha(X) = X^3 + (1/x)X^2 + (1/y)X + (1/xy) \in k(x, y)[X]$. Then α is anti-integral over R of degree 3 (*cf.* Theorems 2.2.9 and 2.2.8. We have $I_{[\alpha]} = xyR$ and hence $I_{[\alpha]}R[\alpha] = xyR[\alpha] \neq R[\alpha]$. It is easy to see that $J_{[\alpha]} = R$. Hence $R[\alpha]$ is flat over R by Theorem 2.3.6.

Example 9.4.3 In view of Example 9.4.14, we know that the implication (ii) $\Rightarrow$ (iii) is not always valid as follows. Let $\beta \in L$ satisfy $\beta^d = a \in R$ and $a \notin U(R)$ with $[K(\beta) : K] = d$ for $d > 1$. Then $\varphi_\beta(X) = X^d - a$ and $\varphi'_\beta(\beta) = d\beta^{d-1} \notin U(R[\beta])$. Thus $R[\beta]$ is integral and hence flat over R by Corollary 2.3.5 but not unramified over R.

Remark 9.4.4 If $I_{[\alpha]} = R$, then $R[\alpha]$ is integral over R by Theorem 2.3.2. In this case, $I_{[\alpha]}R[\alpha] = R[\alpha]$. But in general, we cannot say that $R[\alpha]$ is unramified over R. In fact, when $I_{[\alpha]}R[\alpha] = R[\alpha]$, $R[\alpha]$ is unramified over R if and only if $\varphi'_\alpha(\alpha)R[\alpha] = R[\alpha]$ (*cf.* Theorem 5.1.1). But $\varphi'_\alpha(\alpha)$ is not always a unit in $R[\alpha]$ even if $R[\alpha]$ is integral over R. So the implication (i) $\Rightarrow$ (iii) does not always hold. $\square$

Let α be an element algebraic over R. For an ideal N of R, put $V(N) := \{p \in \mathrm{Spec}(R) | N \subseteq p\}$. Let $\Delta_{R[\alpha]/R} := \{p \in \mathrm{Spec}(R) | pR[\alpha] = R[\alpha]\}$ and $\Gamma_{J_{[\alpha]}} := \{p \in \mathrm{Spec}(R) | p + J_{[\alpha]} = R\}$.

Proposition 9.4.5 *Assume that α is an anti-integral element over R of degree d. Then the following statements are equivalent:*

(1) $I_{[\alpha]}R[\alpha] = R[\alpha]$;

(2) $\sqrt{\tilde{J}_{[\alpha]}} = \sqrt{I_{[\alpha]}}$ *and* $J_{[\alpha]} = R$.

If (2) *holds, then* $\mathrm{Im}[\mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R)] = \mathrm{Spec}(R) \setminus V(I_{[\alpha]})$, *an open subset in* $\mathrm{Spec}(R)$.

Proof (1) $\Rightarrow$ (2): Since $R[\alpha]$ is flat over R by Proposition 9.4.1, we have $J_{[\alpha]} = R$ by Theorem 2.3.6. Thus $\Delta_{R[\alpha]/R} = V(\tilde{J}_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}} = V(\tilde{J}_{[\alpha]})$ (*cf.* Theorem 4.3.2) and hence $I_{[\alpha]} \subseteq \tilde{J}_{[\alpha]}$. So we have $\Delta_{R[\alpha]/R} \subseteq V(I_{[\alpha]})$. Since $I_{[\alpha]}R[\alpha] = R[\alpha]$ by the assumption, we have $V(I_{[\alpha]}) \subseteq \Delta_{R[\alpha]/R}$. Hence $V(I_{[\alpha]}) = \Delta_{R[\alpha]/R} = V(\tilde{J}_{[\alpha]})$, which means that $\sqrt{\tilde{J}_{[\alpha]}} = \sqrt{I_{[\alpha]}}$.
(2) $\Rightarrow$ (1): From the assumption, it follows that $\Delta_{R[\alpha]/R} = V(\tilde{J}_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}} = V(\tilde{J}_{[\alpha]}) = V(I_{[\alpha]})$. Hence $I_{[\alpha]}R[\alpha] = R[\alpha]$.
Next assume (2) holds. We know $\mathrm{Im}[\mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R)] = (\mathrm{Spec}(R) \setminus V(\tilde{J}_{[\alpha]})) \cup V(J_{[\alpha]})$ by Theorem 4.3.2. So $(\mathrm{Spec}(R) \setminus V(\tilde{J}_{[\alpha]})) \cup V(J_{[\alpha]}) = \mathrm{Spec}(R) \setminus V(I_{[\alpha]})$ by (2). $\square$

Remark 9.4.6 Assume that α is anti-integral over R of degree d. If $I_{[\alpha]}R[\alpha] = R[\alpha]$, then $I_{[\alpha]}$ is an invertible ideal of R because $R = J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$ by Proposition 9.4.5. $\square$

Proposition 9.4.7 *Assume that α is an anti-integral element over R of degree d. Consider the following statements:*

(1) $I_{[\alpha]}R[\alpha] = R[\alpha]$;

(2) $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$;

(3) $I_{[\alpha]} = I_{\eta_d}$ *and* $J_{\eta_d} = R$.

Then the implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) *hold.*

Proof (1) $\Rightarrow$ (2): Suppose that $I_{[\alpha]} + I_{[\alpha^{-1}]} \neq R$. Take $p \in \mathrm{Spec}(R)$ such that $I_{[\alpha]} + I_{[\alpha^{-1}]} \subseteq p$. Then $I_{[\alpha]}R_p = aR_p$ for some $a \in I_{[\alpha]}$ by Remark 9.4.6. Put $a_0 = a, a\eta_i = a_i$ $(1 \leq i \leq d)$. Then we have $a_0\alpha^d + a_1\alpha^{d-1} + \cdots + a_d = 0$. Since $I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) = R$ and $\sqrt{\tilde{J}_{[\alpha]}} = \sqrt{I_{[\alpha]}}$ by Proposition 9.4.5, a_d is an invertible element in R_p. Moreover we have $a_0, \cdots, a_{d-1} \in \sqrt{\tilde{J}_{[\alpha]}} = \sqrt{I_{[\alpha]}}$. So we can take a large integer n such that $(a_d)^n = (-(a_0\alpha^d + \cdots + a_{d-1}\alpha))^n = a(b_N\alpha^N + \cdots + b_n\alpha^n)$ for some $b_i \in R_p$. Thus we have an algebraic relation:

$$c_N\alpha^N + \cdots + c_1\alpha + \frac{1}{a} = 0$$

for some $c_i \in R_p$. This means that α is comonic over R_p, i.e., α^{-1} is integral over R_p. Since α is anti-integral over R, α^{-1} is also anti-integral over R. Hence α^{-1} is integral and anti-integral over R_p. So we have $I_{[\alpha^{-1}]}R_p = R_p$, which contradicts the assumption $I_{[\alpha^{-1}]} \subseteq p$.
(2) $\Leftrightarrow$ (3): Since $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$, (2) implies that $R = I_{[\alpha]} + I_{[\alpha^{-1}]} = I_{[\alpha]}(1, \eta_d)$. Hence $I_{[\alpha]} = I_{\eta_d}$ and $J_{\eta_d} = I_{\eta_d}(1, \eta_d) = R$. The converse implication is easy to see. $\square$

Example 9.4.8 Let R be a polynomial ring $k[a, b]$ over a field k and α be a root of the equation:

$$\frac{a}{a-1}X^2 + \frac{b}{a(a-1)}X + \frac{a-1}{a} = 0$$

Then the minimal polynomial of α is given by

$$\varphi_\alpha(X) = X^2 + \frac{b}{a^2}X + \left(\frac{a-1}{a}\right)^2$$

In this case, $I_{[\alpha]} = a^2R$; $\varphi_{\alpha^{-1}}(X) = X^2 + (b/(a-1)^2)X + (a/(a-1))^2$, and $I_{[\alpha^{-1}]} = (a-1)^2R$. Thus $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$, $J_{[\alpha]} = R$ and $\tilde{J}_{[\alpha]} = a^2(1, b/a^2)R = (a^2, b)R$. Hence $\mathrm{grade}(\tilde{J}_{[\alpha]}) > 1$. Therefore $\sqrt{\tilde{J}_{[\alpha]}} \neq \sqrt{I_{[\alpha]}}$, which means that

$I_{[\alpha]}R[\alpha] \neq R[\alpha]$. The implication (2) $\Rightarrow$ (1) in Proposition 9.4.7 is not always valid.

Next, under the condition $I_{[\alpha]}R[\alpha] = R[\alpha]$, we get a condition which is equivalent to the statement that $R[\alpha]$ is unramified over R.

Theorem 9.4.9 *Assume that α is an anti-integral element over R of degree d and that $I_{[\alpha]}R[\alpha] = R[\alpha]$. Then $R[\alpha][\eta_1, \ldots, \eta_d]$ is unramified over $R[\eta_1, \ldots, \eta_d]$ if and only if $R[\alpha]$ is unramified over R.*

Proof ($\Rightarrow$): Take $P \in \mathrm{Spec}(R[\alpha])$ and put $p = P \cap R$. Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, we have $I_{[\alpha]} \not\subseteq p$. Thus $\eta_1, \ldots, \eta_d \in R_p$. Since $R[\alpha][\eta_1, \ldots, \eta_d]$ is unramified over $R[\eta_1, \ldots, \eta_d]$, it follows that $R_p[\alpha] = R_p[\alpha][\eta_1, \ldots, \eta_d]$ is unramified over $R[\eta_1, \ldots, \eta_d] = R_p$, So we see that

$$\begin{aligned}
&I_{[\alpha]}\varphi'_\alpha(\alpha)R_p[\alpha][\eta_1, \ldots, \eta_d] \\
&= I_{[\alpha]}\varphi'_\alpha(\alpha)R_p[\alpha] \\
&= R_p[\alpha] \\
&= R_p[\alpha][\eta_1, \ldots, \eta_d]
\end{aligned}$$

by Lemma 9.3.7. Hence $\varphi'_\alpha(\alpha)$ is a unit in $R_p[\alpha][\eta_1, \ldots, \eta_d] = R_p[\alpha]$ because $I_{[\alpha]} \not\subseteq p$. Therefore $R_p[\alpha]$ is unramified over R_p. So we conclude that $R[\alpha]$ is unramified over R.
($\Leftarrow$): Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, we have only to show that $\varphi'_\alpha(\alpha)$ is a unit in $R[\alpha][\eta_1, \ldots, \eta_d]$ by Lemma 9.3.7. Take $P' \in \mathrm{Spec}(R[\alpha][\eta_1, \ldots, \eta_d])$, and put $P := P' \cap R[\alpha]$ and $p := P \cap R$. Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, we have $p \not\supseteq I_{[\alpha]}$. Hence $\eta_1, \ldots, \eta_d \in R_p$ and $R_p[\alpha][\eta_1, \ldots, \eta_d] = R_p[\alpha]$. Since $R[\alpha]$ is unramified over R by the assumption, it follows that $\varphi'_\alpha(\alpha)^{-1} \in R_p[\alpha]$, which means that $\varphi'_\alpha(\alpha)$ is a unit in $R_p[\alpha]$. Therefore we conclude that $\varphi'_\alpha(\alpha)$ is a unit in $R[\alpha][\eta_1, \ldots, \eta_d]$. $\square$

Remark 9.4.10 Under the same assumptions as in Theorem 9.4.9, $R[\alpha][\eta_1, \ldots, \eta_d]$ is unramified over $R[\eta_1, \ldots, \eta_d]$ if and only if $\varphi'_\alpha(\alpha)$ is a unit in $R[\alpha][\eta_1, \ldots, \eta_d]$. (See Remark 9.4.4.) $\square$

Theorem 9.4.11 *Assume that α is an anti-integral element over R of degree d. Consider the following statements:*

(1) $I_{[\alpha]}R[\alpha] = R[\alpha]$;
(2) $I_{[\alpha]}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$;
(3) $R[\eta_1, \ldots, \eta_d]$ *is flat over* R.

Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold.*

Proof (1) $\Rightarrow$ (2): The assumption $I_{[\alpha]}R[\alpha] = R[\alpha]$ yields $I_{[\alpha]}R[\alpha][\eta_1, \ldots, \eta_d] = R[\alpha][\eta_1, \ldots, \eta_d]$. Since α is integral over $R[\eta_1, \ldots, \eta_d]$ with a monic relation $\varphi_\alpha(X) \in R[\eta_1, \ldots, \eta_d][X]$. So we have $I_{[\alpha]}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$.
(2) $\Rightarrow$ (3): Since $I_{[\alpha]}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$, we have $I_{\eta_i}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$ for each i. Thus $I_{\eta_1} \cdots I_{\eta_d}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$. Hence $R[\eta_1, \ldots, \eta_d]$ is flat over R by Theorem 9.1.1. $\square$

Theorem 9.4.12 *Assume that α is an anti-integral element over R. If $R[\alpha]$ is flat over R, then $R[\eta_1, \ldots, \eta_d]$ is flat over R.*

Proof Since $R[\alpha]$ is flat over R, we have $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)R = R$ by Theorem 2.3.6. Hence $(1, \eta_1, \ldots, \eta_d)R$ is an invertible ideal of R. Localizing at $p \in \mathrm{Spec}(R)$, we can assume that $(1, \eta_1, \ldots, \eta_d)R_p = (1/a)R_p$ for some $a \in R$. Hence $R[\eta_1, \ldots, \eta_d]_p = R_p[1/a]$. Thus $R[\eta_1, \ldots, \eta_d]$ is flat over R. $\square$

Example 9.4.13 In Theorem 9.4.11, the implication (2) $\Rightarrow$ (1) does not always hold. Let $R := k[x, y]$ be a polynomial ring over a field k. Let α be an element satisfying the minimal polynomial $\varphi_\alpha(X) := X^2 + (1/x)X + 1/y \in k(x, y)[X]$. Then α is anti-integral over R. It is easy to see that $I_{[\alpha]} = xyR$ and $J_{[\alpha]} = (x, y)R$. Hence $R[\alpha]$ is not flat over R (Theorem 2.3.6). But both $1/x$ and $1/y$ are strongly flat elements over R. So $R[1/x, 1/y]$ is flat over R. In this case, $I_{[\alpha]}R[1/x, 1/y] = R[1/x, 1/y]$ but $I_{[\alpha]}R[\alpha] \neq R[\alpha]$. Indeed, $J_{[\alpha]} \neq R$ and $I_{[\alpha]}R[\alpha] \subseteq J_{[\alpha]}R[\alpha] \neq R[\alpha]$.

Next we investigate the relationship between $R[\alpha]$ and $R[\eta_1, \ldots, \eta_d]$. Let $\overline{R}$ denote the integral closure of R in K. Then $\overline{R}$ is a Krull domain (*cf.* [M1]).

Theorem 9.4.14 *$R[\alpha]$ is integral over R if and only if $R[\eta_1, \ldots, \eta_d]$ is integral over R.*

Proof ($\Rightarrow$): Since $\overline{R}$ is a Krull domain, α is anti-integral over $\overline{R}$ (*cf.* Theorem 2.2.9). By the assumption, we have α is integral over R, and hence α is integral over $\overline{R}$. Thus $\varphi_\alpha(X) \in \overline{R}[X]$ (*cf.* [M2]). So we have $\eta_1, \ldots, \eta_d$ are integral over R. Therefore $R[\eta_1, \ldots, \eta_d]$ is integral over R.
($\Leftarrow$): Note that α satisfies the monic relation $\varphi_\alpha(X) = 0$ over $R[\eta_1, \ldots, \eta_d]$. Since $R[\eta_1, \ldots, \eta_d]$ is integral over R, α is integral over R. $\square$

Lemma 9.4.15 *Assume that α is a super-primitive element over R. If $I_{[\alpha]}R[\alpha] = R[\alpha]$, then $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$.*

Proof Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, $R[\alpha]$ is flat over R by Proposition 9.4.1. Take $P \in \mathrm{Dp}_1(R[\alpha])(:= \{P \in \mathrm{Spec}(R) | \mathrm{depth}(R_P) = 1\})$ and put $p := P \cap R$. Then $p \in \mathrm{Dp}_1(R)$. Since α is super-primitive over R, we have $J_{[\alpha]}R_p = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)R_p = R_p$. So $I_{[\alpha]}R_p$ is an invertible ideal, and hence $I_{[\alpha]}R_p$ is a principal ideal of R_p. Thus there exists an element $a \in I_{[\alpha]}$ such that $I_{[\alpha]}R_p = aR_p$. Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, it follows that $aR_p[\alpha] = R_p[\alpha]$. So since $I_{[\alpha]} \subseteq I_{\eta_i}$, we have $a\eta_i = b_i$ for some $b_i \in R_p$. Hence $\eta_i \in R_p[\alpha] \subseteq R[\alpha]_P$ because a is a unit in $R_p[\alpha]$. Therefore $\eta_i \in \bigcap_{P \in \mathrm{Dp}_1(R[\alpha]} R[\alpha]_P = R[\alpha]$, which means that $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$. ∎

Theorem 9.4.16 *Assume that α is a super-primitive element over R. Then $I_{[\alpha]}R[\alpha] = R[\alpha]$ if and only if $R[\eta_1, \ldots, \eta_d]$ is flat over R and $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$.*

Proof ($\Rightarrow$): From $I_{[\alpha]}R[\alpha] = R[\alpha]$, it follows that $I_{[\alpha]}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$ and $R[\eta_1, \ldots, \eta_d]$ is flat over R by Theorem 9.4.11. So we have $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$ by Theorem 9.4.15.
($\Leftarrow$): Let $I_{[\alpha]}^{R[\eta_1,\ldots,\eta_d]} := \bigcap_{i=1}^{d} I_{\eta_i}^{R[\eta_1,\ldots,\eta_d]}$, where $I_{\eta_i}^{R[\eta_1,\ldots,\eta_d]} := \{b \in R[\eta_1, \ldots, \eta_d] \mid b\eta_i \in R[\eta_1, \ldots, \eta_d]\}$. Since α is an anti-integral element over R, α is also anti-integral over $R[\eta_1, \ldots, \eta_d]$ and $I_{\eta_i}^{R[\eta_1,\ldots,\eta_d]} = I_{[\alpha]}R[\eta_1, \ldots, \eta_d]$. Since $R[\alpha]$ is integral over $R[\eta_1, \ldots, \eta_d]$ with the integral dependence $\varphi_\alpha(\alpha) = 0$, we have $I_{\eta_i}^{R[\eta_1,\ldots,\eta_d]} = R[\eta_1, \ldots, \eta_d]$. Thus $I_{[\alpha]}R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$, and hence $I_{[\alpha]}R[\alpha] = R[\alpha]$ because $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$. ∎

Corollary 9.4.17 *Assume that α is a super-primitive element over R. If $I_{[\alpha]}R[\alpha] = R[\alpha]$, then $R[\alpha] \cap K = R[\eta_1, \ldots, \eta_d]$.*

Proof Note that $K(\eta_1, \ldots, \eta_d) = K$. By Theorem 9.4.15, $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$. Since $R[\alpha]$ is integral over $R[\eta_1, \ldots, \eta_d]$ and α is anti-integral over $R[\eta_1, \ldots, \eta_d]$, $R[\alpha]$ is free over $R[\eta_1, \ldots, \eta_d]$ of rank d. So $R[\alpha] = R[\eta_1, \ldots, \eta_d] + R[\eta_1, \ldots, \eta_d]\alpha + \cdots + R[\eta_1, \ldots, \eta_d]\alpha^{d-1} \subseteq K + K\alpha + \cdots + K\alpha^{d-1}$, which means that $R[\alpha] \cap K = R[\eta_1, \ldots, \eta_d]$. ∎

Theorem 9.4.18 *Assume that α is a super-primitive element over R of degree d. Then $R[\alpha] \cap K = R$ if and only if $R[\alpha] \cap R[\eta_1, \ldots, \eta_d] = R$.*

Proof The implication ($\Rightarrow$) is obvious because $R[\eta_1, \ldots, \eta_d] \subseteq K$. We must prove the converse implication. Take $a \in \tilde{I}_{[\alpha]}$. Then $a\eta_d = -(a\alpha^d + a\eta_1\alpha^{d-1} + \cdots + a\eta_{d-1}\alpha) \in R[\alpha]$. Thus $\tilde{I}_{[\alpha]}\eta_d \subseteq R[\alpha] \cap R[\eta_1, \ldots, \eta_d] = R$, which implies that $\tilde{I}_{[\alpha]} \subseteq I_{\eta_d}$. So we conclude that α is exclusive by Corollary 7.1.3. ∎

Proposition 9.4.19 *If there exists $p \in \mathrm{Spec}(R)$ such that $R[\eta_1, \ldots, \eta_d]_p$ is integral over R_p and $R[\eta_1, \ldots, \eta_d]_p \neq R_p$, then α is not anti-integral over R.*

Proof Suppose that α is anti-integral over R. Then α is anti-integral over R_p. Since $R[\eta_1, \ldots, \eta_d]_p$ is integral over R_p, it follows that α is integral over R_p by Theorem 9.4.14. Thus $\varphi_\alpha(X) \in R_p[X]$ (*cf.* Theorem 9.3.2), which contradicts the assumption $R[\eta_1, \ldots, \eta_d]_p \neq R_p$. ∎

Recall that $\mathcal{C}(\overline{R}/R)$ denotes the conductor between R and $\overline{R}$.

Proposition 9.4.20 *Assume that $\overline{R}$ is a finite R-module. If there exists a prime divisor p of $I_{[\alpha]}$ such that $p \not\supseteq I_{[\alpha]} :_R \mathcal{C}(\overline{R}/R)$, then α is not anti-integral over R.*

Proof Localizing at p, we may assume that (R, p) is a local ring with $p \in \mathrm{Dp}_1(R)$. We have $\mathcal{C}(\overline{R}/R) \subseteq I_{[\alpha]} \subseteq I_{\eta_i}$. Thus $\eta_i \in \mathcal{C}(\overline{R}/R)^{-1} = \overline{R}$ for $1 \leq i \leq d$. But since $p \supseteq I_{[\alpha]}$, we have $R[\eta_1, \ldots, \eta_d] \neq R$. Our conclusion follows from Proposition 9.4.19. ∎

Remark 9.4.21 If α is ultra-primitive over R, i.e., $\mathrm{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$, then α is an anti-integral element over R. Thus "ultra-primitive $\Rightarrow$ super-primitive $\Rightarrow$ anti-integral" (*cf.* Proposition 7.5.1 and Theorem 2.2.8). ∎

We close this section by giving the following result.

Proposition 9.4.22 *Assume that α is an anti-integral element over R of degree d. Let $\beta_1, \cdots, \beta_n$ be elements in K, each of which is a super-primitive element over R. If $R[\alpha]$ is flat over R, then $\beta_1, \cdots, \beta_n$ are super-primitive elements over $R[\alpha]$.*

Proof Note that $I_{[\beta_i]} = I_{\beta_i} (:= R :_R \beta_i)$ and $J_{[\beta_i]} = I_{\beta_i} + \beta_i I_{\beta_i} = I_{\beta_i} + I_{\beta_i^{-1}}$ because $\beta_i \in K$. Since β_i is super-primitive over R, we have $\mathrm{grade}(I_{\beta_i} + I_{\beta_i^{-1}}) = \mathrm{grade}(J_{[\beta_i]}) > 1$ by definition. Let $T_{\beta_i} := R[\alpha] :_R \beta_i$. Since $R[\alpha]$ is flat over R, we have $I_{\beta_i} + I_{\beta_i^{-1}} \subseteq T_{\beta_i} + T_{\beta_i^{-1}}$, and hence we have $\mathrm{grade}(T_{\beta_i} + T_{\beta_i^{-1}}) > 1$. Thus β_i is super-primitive over $R[\alpha]$ by Proposition 9.4.12. ∎

9.5 Contractions of Principal Ideals and Denominator Ideals

Let R be a Noetherian domain, $R[X]$ be a polynomial ring, α be an element of an algebraic extension field of the quotient field K of R, and let $A := R[\alpha]$.

Our main aims of this section are to investigate the contraction $\beta A \cap R$ of a principal fractional ideal βA with an element $\beta \in K(\alpha) = K[\alpha]$, the contraction $(A :_A \beta) \cap R$ of a denominator ideal $A :_A \beta$, and to show what conditions are required for $\beta A = A$ or $\beta A \cap R = R$.

In what follows, we use the following notations unless otherwise specified:

Let R be a Noetherian domain (which is commutative and has a unit), $R[X]$ be a polynomial ring, α be an element of an algebraic extension field of the quotient field K of R, and $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Then $\eta_i \in K$ $(1 \le i \le d)$ are uniquely determined by α.

Put

$$I_{\eta_i} := R :_R \eta_i,$$

$$I_{[\alpha]} := \bigcap_{i=1}^{d} I_{\eta_i},$$

$$E^{(\alpha)} := Ker(\pi)\text{, and}$$

$$E_K^{(\alpha)} := E^{(\alpha)} \otimes_R K = \varphi_\alpha(X)K[X].$$

For an element γ in $K[\alpha]$, we can write $\gamma = \tau_0 + \tau_1\alpha + \cdots + \tau_{d-1}\alpha^{d-1}$ uniquely with $\tau_i \in K$. Put

$$\gamma_*(X) := \tau_0 + \tau_1 X + \cdots + \tau_{d-1}X^{d-1} \in K[X] \text{ and}$$

$$\Lambda_\alpha(\gamma) := E_K^{(\alpha)} + \gamma_*(X).$$

The ideal $I_{[\alpha]}$ is also expressed as $R[X] :_R \varphi_\alpha(X) := \{a \in R | a\varphi_\alpha(X) \in R[X]\}$ and $I_{[\alpha]}\varphi_\alpha(X)R[X] \subseteq E^{(\alpha)}$. Since π (respectively $\pi \otimes_R K$) is surjective, every element in A (respectively $K[\alpha]$) is expressed as $f(\alpha)$ for some $f(X)$ in $R[X]$ (respectively $K[X]$). We have $\gamma_*(\alpha) = \gamma \in L$ and $\Lambda_\alpha(\gamma) = \{\psi(X) \in K[X] | \psi(\alpha) = \gamma\}$. For any $\phi(X) \in \Lambda_\alpha(\gamma)$, $\Lambda_\alpha(\gamma) = E_K^{(\alpha)} + \phi(X)$. For $\gamma \in L \setminus \{0\}$, we have that $\gamma \in A \Leftrightarrow \Lambda_\alpha(\gamma) \cap R[X] \neq \emptyset$ and $\gamma \in L \setminus A \Leftrightarrow \Lambda_\alpha(\gamma) \cap R[X] = \emptyset$.

Lemma 9.5.1 *Let $\beta \in L = K[\alpha]$. Then $\beta A \cap R = (E_K^{(\alpha)} + \psi(X)R[X]) \cap R$ for every $\psi(X) \in \Lambda_\alpha(\beta)$.*

Proof Take $a \in \beta A \cap R$. Then $a = \beta g(\alpha)$ for some $g(\alpha) \in A$. Since $a - \beta g(\alpha) = a - \psi(\alpha)g(\alpha) = 0$, we have $a - \psi(X)g(X) \in E_K^{(\alpha)} = \varphi_\alpha(X)K[X]$. Thus $a - \psi(X)g(X) = \varphi_\alpha(X)\tau(X)$ for some $\tau(X) \in K[X]$, which means that $a = \varphi_\alpha(X)\tau(X) + \psi(X)g(X) \in (E_K^{(\alpha)} + \psi(X)R[X]) \cap R$. Thus $\beta A \cap R \subseteq (E_K^{(\alpha)} + \psi(X)R[X]) \cap R$. Conversely, take $a \in (E_K^{(\alpha)} + \psi(X)R[X]) \cap R$ with

$\psi(X) \in \Lambda_\alpha(\beta)$. Then $a = \varphi_\alpha(X)\tau(X) + \psi(X)g(X)$ for some $\tau(X) \in K[X]$ and $g(X) \in R[X]$. Hence $a = \psi(\alpha)g(\alpha) = \beta g(\alpha) \in \beta A \cap R$. So we have $(E_K^{(\alpha)} + \psi(X)R[X]) \cap R \subseteq \beta A \cap R$. Therefore we have $\beta A \cap R = (E_K^{(\alpha)} + \psi(X)R[X]) \cap R$. ∎

Corollary 9.5.2 *Assume that $\beta \in A$. Then $\Lambda_\alpha(\beta) \cap R[X] \neq \emptyset$ and $\beta A \cap R = (E^{(\alpha)}, f(X))R[X] \cap R$ for every $f(X) \in \Lambda_\alpha(\beta) \cap R[X]$.*

Proof We have seen that $\Lambda_\alpha(\beta) \cap R[X] \neq \emptyset$ because $\beta \in A$. We have that $(E_K^{(\alpha)} + f(X)R[X]) \cap R = \beta A \cap R$ for every $f(X) \in \Lambda_\alpha(\beta)$ by Lemma 9.5.1. Take $f(X) \in \Lambda_\alpha(\beta) \cap R[X] \neq \emptyset$. Then $\beta A \cap R = (E_K^{(\alpha)} + f(X)R[X]) \cap R = (E_K^{(\alpha)} + f(X)R[X]) \cap R[X] \cap R = ((E_K^{(\alpha)} \cap R[X]) + f(X)R[X]) \cap R = (E^{(\alpha)}, f(X))R[X] \cap R$. ∎

Lemma 9.5.3 *Assume that $\beta \neq 0$ in L. Then there exist $\psi_1(X)$ and $\psi_2(X)$ in $K[X]$ such that $\varphi_\alpha(X)\psi_1(X) + \beta_*(X)\psi_2(X) = 1$ in $K[X]$ and that* $\deg \psi_2 < d$. *Furthermore, such polynomials $\psi_1(X)$ and $\psi_2(X)$ are uniquely determined.*

Proof Since $\varphi_\alpha(X)$ is irreducible polynomial in $K[X]$ and since $\beta \neq 0$, we have $(\varphi_\alpha(X), \beta_*(X))K[X] = K[X]$. So there exist $\psi_1(X), \psi_2(X) \in K[X]$ such that $\varphi_\alpha(X)\psi_1(X) + \beta_*(X)\psi_2(X) = 1$. Since $\varphi_\alpha(X)$ is a monic polynomial in $K[X]$, we can take $\psi_2(X)$ such as $\deg \psi_2(X) < d$. Next we show that $\psi_1(X)$ and $\psi_2(X)$ are uniquely determined. Suppose that there exist $\lambda_1(X), \lambda_2(X) \in K[X]$ which satisfy the same conditions as for $\psi_1(X), \psi_2(X)$. Then $\varphi_\alpha(X)\psi_1(X) + \beta_*(X)\psi_2(X) = 1$ and $\varphi_\alpha(X)\lambda_1(X) + \beta_*(X)\lambda_2(X) = 1$. Hence $\varphi_\alpha(X)(\psi_1(X) - \lambda_1(X)) + \beta_*(X)(\psi_2(X) - \lambda_2(X)) = 0$. Since $(\varphi_\alpha(X), \beta_*(X))K[X] = K[X]$, we have $\psi_2(X) - \lambda_2(X) = \varphi_\alpha(X)\mu(X)$ for some $\mu(X) \in K[X]$. We see that $\deg \psi_2(X) < d$ and $\deg \lambda_2(X) < d$ imply $\psi_2(X) - \lambda_2(X) = 0$, that is, $\psi_2(X) = \lambda_2(X)$. Thus we have also that $\psi_1(X) = \lambda_1(X)$. ∎

We have $[K(\alpha) : K] = d$ and $L = K \oplus K\alpha \oplus \cdots \oplus K\alpha^{d-1}$. Since $\varphi_\alpha(\alpha) = 0$, we have $\psi_2(\alpha) = \beta^{-1} \in L$. So we write: $\beta^{-1} = \beta_*(\alpha)^{-1} = \lambda_0 + \lambda_1\alpha + \cdots + \lambda_{d-1}\alpha^{d-1}$. Noting that $\deg \psi_2(X) < d$, we have $\psi_2(X) = \lambda_0 + \lambda_1 X + \cdots + \lambda_{d-1}X^{d-1} \in K[X]$.

For $\psi(X) \in K[X]$, let $I^R_{\psi(X)} := \{a \in R | a\psi(X) \in R[X]\}$.

Proposition 9.5.4 *Under the same notation as in Lemma* 9.5.3. *If $I_{[\alpha]} = R$, i.e., $\varphi_\alpha(X) \in R[X]$, then $\beta A \cap R = I^R_{\psi_2(X)}$.*

Proof Take $a \in I^R_{\psi_2(X)}$. Then $a = \varphi_\alpha(X)(a\psi_1(X)) + \beta_*(X)(a\psi_2(X)) \in (E_K^{(\alpha)} + \beta_*(X)R[X]) \cap R = \beta A \cap R$ by Lemma 9.5.1. Conversely, take $a \in \beta A \cap R$.

Then $a = \varphi_\alpha(X)\psi(X) + \beta_*(X)h(X)$ for some $\psi(X) \in K[X]$ and some $h(X) \in R[X]$ by Lemma 9.5.1. Since $\varphi_\alpha(X)$ is a monic polynomial in $R[X]$, we can assume that $\deg h(X) < d$. So we have $1 = \varphi_\alpha(X)(a^{-1}g(X)) + \beta_*(X)(a^{-1}h(X))$. Thus $a^{-1}h(X) = \psi_2(X)$ by Lemma 9.5.3, which means that $a \in I^R_{\psi_2(X)}$. ∎

Remark 9.5.5 [OY15] For $\beta \in K(\alpha)$, $\beta R[\alpha] \supseteq R[\alpha] \Leftrightarrow \beta^{-1} \in R[\alpha] \Leftrightarrow \beta R[\alpha] \cap R = R$. When $\beta \in R[\alpha]$, $\beta R[\alpha] = R[\alpha] \Leftrightarrow \beta R[\alpha] \cap R = R$. ∎

Corollary 9.5.6 *Assume that $\beta \neq 0$ in L and that $\varphi_\alpha(X) \in R[X]$. Then $\beta R[\alpha] \cap R = R$ if and only if $\psi_2(X) \in R[X]$. Assume furthermore that $\beta \in R[\alpha]$. Then $\beta R[\alpha] = R[\alpha]$ if and only if $\psi_2(X) \in R[X]$.*

Proof By Proposition 9.5.4, $\beta R[\alpha] \cap R = R \Leftrightarrow I^R_{\psi_2(X)} = R \Leftrightarrow \psi_2(X) \in R[X]$. Next, when $\beta \in R[\alpha]$, $\beta R[\alpha] \cap R = R \Leftrightarrow \beta R[\alpha] = R[\alpha$ by Remark 9.5.5. ∎

Definition 9.5.7 Let $\beta \in K(\alpha) \setminus \{0\}$ and $\beta^{-1} = \lambda_0 + \lambda_1\alpha + \cdots + \lambda_{d-1}\alpha^{d-1} \in K(\alpha) = K \oplus K\alpha \oplus \cdots \oplus K\alpha^{d-1}$. We define:

$$H_{[\beta]} := \bigcap_{i=0}^{d-1} I_{\lambda_i}$$

Note that $\deg \psi_2(X) < d$, $\beta_*^{-1}(X) = \psi_2(X) \in K[X]$, and $\psi_2(X) = \lambda_0 + \lambda_1 X + \cdots + \lambda_{d-1}X^{d-1}$ as the same notation as before.

Remark 9.5.8 It is easy to see that $H_{[\beta]} = I^R_{\psi_2(X)}$ for $\beta \in K(\alpha) \setminus \{0\}$ by definition. Hence for $\beta \in K(\alpha) \setminus \{0\}$, $\beta R[\alpha] \cap R = H_{[\beta]}$ by Proposition 9.5.4 if $I_{[\alpha]} = R$. ∎

Remark 9.5.9

(1) For $\lambda \in K$, $I_\lambda := R :_R \lambda$ is divisorial. In fact, it is easy to see that $I_\lambda = \lambda^{-1}R \cap R$, which is the intersection of fractional principal ideals. So I_λ is divisorial by [F, p.12]. Thus the ideal $H_{[\beta]}$ is a divisorial ideal of R.

(2) If I be a nontrivial divisorial ideal of R, then grade$(I) = 1$. Indeed, suppose that grade $(I) > 1$. Then I contains a regular sequence a, b. Consider $I^{-1} := R :_K I$. Take $\gamma \in I^{-1}$. Then $\gamma a, \gamma b \in R$ and hence $a, b \in \gamma^{-1}R \cap R$. Put $\gamma := c/d$ with $c, d \in R$. Then $ac, bc \in dR$.

So $ac = de, bc = df$ for some $e, f \in R$. Thus $acf = def = bce$ and hence $af = be$. Since a, b is a regular sequence, it follows that $e \in aR$. Let $e = ag$ with $sg \in R$. Then $af = bag$ and so $f = bg$ and $bc = df = dbg$, which implies that $c = dg$. Thus $\gamma = c/d = dg/d = g \in R$. So we have $I^{-1} \subseteq R$, which induces $(I^{-1})^{-1} \supseteq R \supset I$. This contradicts the assumption that I is divisorial, i.e., $(I^{-1})^{-1} = I$.

The following lemma follows from the definition. ∎

Lemma 9.5.10 *Under the above notation, we have*

$$\beta R[\alpha] \cap R = \{a \in R | a\psi_2(\alpha) \in R[\alpha]\} = R[\alpha] :_R \beta^{-1}$$

The following result is a strong version of Proposition 9.5.4 (Remark 9.5.8).

Theorem 9.5.11 *Let β be a nonzero element in L. If* $\operatorname{grade}(H_{[\beta]} + I_{[\alpha]}) > 1$, *then $\beta R[\alpha] \cap R = H_{[\beta]}$.*

Proof Let p be a prime divisor of the ideal $H_{[\beta]}$. Since $H_{[\beta]}$ is a divisorial ideal by Remark 9.5.9(1), we have $\operatorname{depth}(R_p) = 1$ by Remark 9.5.9(2). So by the assumption, we see that $I_{[\alpha]} \not\subseteq p$. Localizing at p, we have $\varphi_\alpha(X) \in R_p[X]$. Thus $(\beta R[\alpha] \cap R)_p \subseteq H_{[\beta]}R_p$ by Proposition 9.5.4 and Remark 9.5.8, and hence $\beta R[\alpha] \cap R \subseteq H_{[\beta]}$. Conversely, take $a \in H_{[\beta]}$. Then $a\lambda_i \in R$ for all i, and hence $a\psi_2(\alpha) \in R[\alpha]$. Thus $a \in \beta R[\alpha] \cap R$ by Lemma 9.5.10. ∎

Remark 9.5.12 Let $\beta \in K(\alpha) \setminus \{0\}$. Then $\beta R[\alpha] \cap R = (R[\alpha] :_{R[\alpha]} \beta^{-1}) \cap R = R[\alpha] :_R \beta^{-1}$. ∎

Corollary 9.5.13 *Let β be an element of $K(\alpha) \setminus \{0\}$. If* $\operatorname{grade}(H_{[\beta^{-1}]} + I_{[\alpha]}) > 1$, *then $(R[\alpha] :_{R[\alpha]} \beta) \cap R = R[\alpha] :_R \beta = H_{[\beta^{-1}]}$.*

Proof Replacing β by β^{-1} in Theorem 9.5.11, we have $\beta^{-1}R[\alpha] \cap R = H_{[\beta^{-1}]}$. Since $\beta^{-1}R[\alpha] \cap R = (R[\alpha] :_{R[\alpha]} \beta) \cap R = R[\alpha] :_R \beta$ by Remark 9.5.12, we come to the conclusion. ∎

Corollary 9.5.14 *Let β be a nonzero element in $K(\alpha)$. Assume that*

$$\operatorname{grade}(H_{[\beta]} + I_{[\alpha]}) > 1$$

Then $\beta R[\alpha] \cap R = R$ if and only if $\psi_2(X) \in R[X]$. Assume furthermore that $\beta \in R[\alpha]$. Then $\beta R[\alpha] = R[\alpha]$ if and only if $\psi_2(X) \in R[X]$.

We now give an example which shows that the condition on "grade" in Theorem 9.5.11 is effective.

Example 9.5.15 Let k be a field and $R := k[t]$ be a polynomial ring whose quotient field is denoted by K. Let α be an element $1/t$. Then $\varphi_\alpha(X) = X - 1/t$. It is easy to see that $I_{[\alpha]} = tR$. Put $\beta = \alpha^{-1} = t$. Then $\beta^{-1} = 1/t$. We have that $H_{[\beta]} = I_{[\alpha]} = tR \neq R$ and $\mathrm{grade}(H_{[\beta]} + I_{[\alpha]}) = \mathrm{grade}(I_{[\alpha]}) = 1$. In this case, we have $\beta R[\alpha] \cap R = tR[1/t] \cap R \supseteq R$. Hence $\beta R[\alpha] \cap R = R \neq H_{[\beta]}$.

Proposition 9.5.16 *If* $\mathrm{grade}(I_{[\alpha]} + I_{[\alpha^{-1}]}) > 1$, *then* $\alpha R[\alpha] \cap R = I_{[\alpha^{-1}]}$.

Proof Since $\varphi_\alpha(\alpha) = \alpha^d + \eta_\alpha^{d-1} + \cdots + \eta_d = 0$, we have $\alpha^{-1} = -(\eta_d^{-1}\alpha^{d-1} + \eta_d^{-1}\eta_1\alpha^{d-2} + \cdots + \eta_d^{-1}\eta_{d-1})$. So $I_{[\alpha^{-1}]} = I_{\eta_d^{-1}} \cap I_{\eta_d^{-1}\eta_1} \cap \cdots \cap I_{\eta_d^{-1}\eta_{d-1}}$. On the other hand, we have $(\alpha^{-1})^d + \eta_d^{-1}\eta_{d-1}(\alpha^{-1})^{d-1} + \cdots + \eta_d^{-1}\eta_1(\alpha^{-1}) + \eta_d^{-1} = 0$. Hence $H_{[\alpha]} = I_{\eta_d^{-1}} \cap I_{\eta_d^{-1}\eta_1} \cap \cdots \cap I_{\eta_d^{-1}\eta_{d-1}} = I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$. Thus our conclusion follows from Theorem 9.5.11. $\square$

Corollary 9.5.17 *Let* $a \in R$ *and assume that* $\mathrm{grade}(aR + I_{[\alpha]}) > 1$. *Then* $aR[\alpha] \cap R = aR$.

Let $\{p_1, \ldots, p_n\}$ be the set of prime divisors of $I_{[\alpha]}$ and $T := R \setminus \bigcup_{i=1}^n p_i$. Then T is a multiplicatively closed subset of R. Let R_T denote the localization of R with respect to T.

Proposition 9.5.18 *Assume that* α *is a super-primitive element of degree* d *over* R *and take* $\lambda \in K$. *Then* $\mathrm{grade}(I_\lambda + I_{[\alpha]}) > 1$ *if and only if* $\lambda \in R_T$.

Proof ($\Leftarrow$): Since $\lambda \in R_T$, there exists $a \in T$ such that $a\lambda \in R$. Hence $a \in I_\lambda$. By the definition of T, we have $a \notin p_1 \cup \cdots \cup p_n$. Let p be a depth 1 prime ideal containing $I_{[\alpha]}$. Since α is super-primitive over R, $(I_{[\alpha]})_p$ is a principal ideal in R_p. Hence p is a prime divisor of $I_{[\alpha]}$. Thus $p = p_i$ for some i. We conclude that $\mathrm{grade}(I_\lambda + I_{[\alpha]}) > 1$. ($\Rightarrow$): Let p_i $(1 \leq i \leq n)$ be a prime divisor of $I_{[\alpha]}$. Then p_i is a depth 1 prime ideal of R because $I_{[\alpha]}$ is a divisorial ideal. Hence $I_\lambda \not\subseteq p_i$ for all i by the assumption. Thus there exists an element $a \in I_{[\alpha]} \cap T$, which means that $\lambda \in R_T$. $\square$

Remark 9.5.19 Put $U := R_T + R_T\alpha + \cdots + R_T\alpha^{d-1}$. Note that U is not always a ring. Let β be an element in L.

(1) $\mathrm{grade}(H_{[\beta]} + I_{[\alpha]}) > 1$ if and only if $\beta^{-1} \in U$. In fact, putting $\beta^{-1} := \lambda_0 + \lambda_1\alpha + \cdots + \lambda_{d-1}\alpha^{d-1}$ with $\lambda_i \in K$. Then $H_{[\beta]} = \bigcap_{i=0}^{d-1} I_{\lambda_i}$. So if $\mathrm{grade}(H_{[\beta]} + I_{[\alpha]}) > 1$, then $H_{[\beta]} \cap T \neq \emptyset$ and hence $\lambda_i \in R_T$ for all i. Thus $\beta^{-1} \in U$. Conversely, if $\beta^{-1} \in U$, then $\lambda_i \in R_T$ for all i. Hence $H_{[\beta]}R_T = R_T$. Since $I_{[\alpha]}$ is a divisorial ideal, we conclude that $\mathrm{grade}(H_{[\beta]} + I_{[\alpha]}) > 1$.

(2) $R[\alpha] \subseteq U$ if and only if $I_{[\alpha]} = R$ and, in this case, U is a ring. Since "if" part is obvious, we have only to show "only if" part. Since $\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0$, we have $-\alpha^d = \eta_1\alpha^{d-1} + \cdots + \eta_d \in A \subseteq R[\alpha] \subseteq U$, which yields that $-\alpha^d = \lambda_1\alpha^{d-1} + \cdots + \lambda_d$ with $\lambda_i \in R_T$. Hence $\eta_1\alpha^{d-1} + \cdots + \eta_d = \lambda_1\alpha^{d-1} + \cdots + \lambda_d$. Since $[K(\alpha) : K] = d$, we have $\lambda_i = \eta_i$ for all i and hence $\eta_i \in R_T$. But we know that $I_{\eta_i} \supseteq I_{[\alpha]}$ and either $\mathrm{grade}(I_{\eta_i}) = 1$ or $I_{\eta_i} = R$. So $I_{\eta_i} = R$ (Proposition 9.5.18). Thus $I_{[\alpha]} = R$. It follows that $\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0$ with $\eta_i \in R_T$, which means that U is a ring. ∎

Proposition 9.5.20 *Let α be an algebraic element of degree d over R and let β be a nonzero element in $R\langle\alpha\rangle$. Then*

$$\beta R\langle\alpha\rangle \cap R = R\langle\alpha\rangle :_R \beta^{-1}$$

Proof Take $a \in R\langle\alpha\rangle \cap R$. Then $a = \beta y$ with some $y \in R\langle\alpha\rangle$. Hence $y \in a\beta^{-1} \in R\langle\alpha\rangle$. Thus $a \in R\langle\alpha\rangle : R\beta^{-1}$. Conversely, take $x \in R$ such that $x\beta^{-1} \in R\langle\alpha\rangle$. Then $x \in \beta R\langle\alpha\rangle \cap R$. ∎

Remark 9.5.21 Under the same notation as in Proposition 9.5.20, it is easy to see that $\gamma R[\alpha] \cap R = R[\alpha] :_R \gamma^{-1}$ for a nonzero element $\gamma \in R[\alpha]$. ∎

Now recall the following structure theorem of the subring $R\langle\alpha\rangle := R[\alpha] \cap R[\alpha^{-1}]$.

Lemma 9.5.22 *(cf. Theorem 3.1.18) Put $\zeta_i := \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i$ for $1 \leq i \leq d-1$ and $\eta_i \in K$. Assume that α is an anti-integral element of degree d over R. Then $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$.*

Corollary 9.5.23 *Assume that α is an anti-integral element of degree d over R. Then $R\langle\alpha\rangle \cap K = R$ (i.e., $R\langle\alpha\rangle$ is exclusive over R).*

Remark 9.5.24 Take $\beta \in R[\alpha]$. Note that $K(\alpha) = K[\alpha] = K + K\alpha + \cdots + K\alpha^{d-1}$, and put $\beta^{-1} := \lambda_0 + \lambda_1\alpha + \cdots + \lambda_{d-1}\alpha^{d-1}$ with $\lambda_i \in K$. We showed

in Theorem 9.5.11, that if $\mathrm{grade}(\bigcap_{i=0}^{d-1} I_{\lambda_i} + I_{[\alpha]}) > 1$ (of course, $I_{[\alpha]} = R$ is sufficient), then $\beta R[\alpha] \cap R = \bigcap_{i=0}^{d-1} I_{\lambda_i}$. ∎

The following theorem is similar to the known result (*cf.* Remark 9.5.24). But as for the extension $R\langle\alpha\rangle$ of R, we need not require the assumption such as $\mathrm{grade}(\bigcap_{i=0}^{d-1} I_{\lambda_i} + I_{[\alpha]}) > 1$.

Let ζ_i be the same as in Lemma 9.5.22. Then $K[\alpha] = K(\alpha) = K(\zeta_1, \ldots, \zeta_{d-1}) = K[\zeta_1, \ldots, \zeta_{d-1}]$.

Theorem 9.5.25 *Assume that α is an anti-integral element of degree d over R. Let β be a nonzero element in $R\langle\alpha\rangle$ and let $\beta^{-1} := \lambda_0 + \lambda_1\zeta_1 + \cdots + \lambda_{d-1}\zeta_{d-1}$ with $\lambda_i \in K$. Then*

$$\beta R\langle\alpha\rangle \cap R = I_{\lambda_0} \cap \bigcap_{i=1}^{d-1} (I_{[\alpha]} :_R \lambda_i)$$

Proof Since α is super-primitive over R, $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 + \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ by Lemma 9.5.22. Hence using Proposition 9.5.20, we have $\beta R\langle\alpha\rangle \cap R = R\langle\alpha\rangle :_R \beta^{-1} = (R \oplus I_{[\alpha]}\zeta_1 + \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}) :_R (\lambda_0 + \lambda_1\zeta_1 + \cdots + \lambda_{d-1}\zeta_{d-1}) = I_{\lambda_0} \cap \bigcap_{i=1}^{d-1} (I_{[\alpha]} :_R \lambda_i)$. ∎

Lemma 9.5.26 *Assume that α is an anti-integral element of degree d over R. If $a, b \in R$ is a regular sequence on R, then it also is $R\langle\alpha\rangle$.*

Proof Take $P \in \mathrm{Dp}_1(R\langle\alpha\rangle)$ such that $a, b \in P$. Then $a, b \in p := P \cap R$. But it is known that $p \in \mathrm{Dp}_1(R)$ (*cf.* Lemma 9.5.22), which is a contradiction. ∎

Proposition 9.5.27 *Assume that α is an anti-integral element of degree d over R. Let $\beta_1, \beta_2 \in R\langle\alpha\rangle$. Assume also that* $\mathrm{grade}(I_{[\alpha]} + (\beta_i R\langle\alpha\rangle \cap R)) > 1$ *for $i = 1, 2$. If* $\mathrm{grade}((\beta_1 R\langle\alpha\rangle \cap R) + (\beta_2 R\langle\alpha\rangle \cap R)) > 1$, *then β_1, β_2 form a regular sequence on $R\langle\alpha\rangle$.*

Proof Take $P \in \mathrm{Dp}_1(R\langle\alpha\rangle)$ such that $\beta_1, \beta_2 \in P$. Put $p := P \cap R$. Then $p \in \mathrm{Dp}_1(R)$ by Lemma 9.5.26. By construction, we have $p \supseteq (\beta_1 R\langle\alpha\rangle \cap R) + (\beta_2 R\langle\alpha\rangle \cap R)$, which shows that $\mathrm{grade}(p) \geq 2$, a contradiction. ∎

Theorem 9.5.28 *Assume that α is an anti-integral element of degree d over R. Let $\beta_1, \ldots, \beta_k \in R\langle\alpha\rangle$. Assume further that $J_{[\alpha]} + \sum_{i=1}^{k} (\beta_i R\langle\alpha\rangle \cap R) = R$. If* $\mathrm{grade}(\sum_{i=1}^{k} (\beta_i R\langle\alpha\rangle \cap R)) \geq n$, *then* $\mathrm{grade}((\beta_1, \ldots, \beta_k)R\langle\alpha\rangle) \geq n$.

Proof Take $P \in \mathrm{Spec}(R\langle\alpha\rangle)$ such that $(\beta_1, \ldots, beta_k) \subseteq P$. Put $p := P \cap R$. Then $J_{[\alpha]} \not\subseteq P$ by the assumption. So $R_p[\alpha]$ is flat over R_p and hence $(R\langle\alpha\rangle)_p$ is flat over R_p, for $(J_{[\alpha]})_p = R_p$ implies that $(I_{[\alpha]})_p$ is an invertible ideal of R_p and $(R\langle\alpha\rangle)_p = R_p \oplus (I_{[\alpha]})_p\zeta_1 \oplus \cdots \oplus (I_{[\alpha]})_p\zeta_{d-1}$ by Lemma 9.5.22. Thus the flatness guarantees that the grade does not decrease. $\square$

Chapter 10

Anti-Integral Ideals and Super-Primitive Polynomials

Let R be a Noetherian integral domain, $R[X]$ be a polynomial ring, and K be the quotient field of R. Let α be an element of an algebraic field extension L of K and $\pi : R[X] \longrightarrow R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Let $I_{[\alpha]} := \bigcap_{i=1}^{d}(R :_R \eta_i)$ $(= R[X] :_R \varphi(X))$. For $f(X) \in R[X]$, let $\mathsf{c}(f(X))$ denote the ideal generated by the coefficients of $f(X)$. Let $J_{[\alpha]} := I_{[\alpha]}\mathsf{c}(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called an anti-integral element of degree d over R if $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, $R[\alpha]$ is called an anti-integral extension of R. In the case $K(\alpha) = K$, an anti-integral element α is the same as an anti-integral element (i.e., $R = R\langle\alpha\rangle$) defined in Chapter 1. The element α is called a super-primitive element of degree d over R if $J_{[\alpha]} \not\subset p$ for all primes p of depth 1.

As was seen in the above, Chapter 3 concerned a simple extension with certain properties by use of the ideal $\mathrm{Ker}(\pi)$. This chapter deals with an ideal H of $R[X]$ such that $P \cap R = (0)$ for all $P \in Ass_{R[X]}(R[X]/H)$ (i.e., exclusive) or a monic polynomial $\varphi(X)$ in $K[X]$. We define the super-primitiveness, anti-integrality and flatness of the ideal H or a polynomial $\varphi(X)$. If necessary, we can consider a simple ring extension $R[X]/H$ or $R[X]/(\varphi(X)K[X] \cap R[X])$. When H is a prime ideal or $\varphi(X) \in K[X]$ is an irreducible polynomial, we come back to the case in Chapter 2. When H is not a prime ideal or $\varphi(X) \in K[X]$ is not an irreducible polynomial, we can extend the super-primitiveness, anti-integrality and flatness to a simple extension which is not necessarily an integral domain.

We use the following notations throughout this section unless otherwise specified:

Let R be a Noetherian integral domain, $R[X]$ be a polynomial ring, and K be the quotient field of R. Let H be an ideal of $R[X]$ and $\varphi(X)$ be a monic polynomial in $K[X]$.

10.1 Anti-Integral Ideals and Super-Primitive Ideals

Let $R[X]$ be a polynomial ring over R and H denote an ideal of $R[X]$. We say that H is an *exclusive ideal* if $P \cap R = (0)$ for every prime ideal $P \in \mathrm{Ass}_{R[X]}(R[X]/H)$. When H is exclusive, $H_K := H \otimes_R K \underset{\neq}{\subset} K[X]$. Since H_K is a principal ideal of $K[X]$, we can write $H_K = \varphi_H(X)K[X]$ for some monic polynomial $\varphi_H(X) \in K[X]$. Note that the monic polynomial $\varphi_H(X)$ is uniquely determined by H. Let $d = \deg \varphi_H(X)$ and put $\varphi_H(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ with $\eta_i \in K$. Let $I_H := \bigcap_{i=1}^{d} I_{\eta_i} (= R[X] :_R \varphi_H(X))$ and $J_H := \mathsf{c}(I_H\varphi_H(X))$. Then $I_H\varphi_H(X)R[X] \subseteq R[X]$. For $f(X) \in K[X]$, $\mathsf{c}(f(X))$ denotes the content ideal of $f(X)$, the fractional ideal of R generated by the coefficients of $f(X)$.

Definition 10.1.1 Let H be an exclusive ideal of $R[X]$.

(1) The ideal H is called an *anti-integral ideal* of $R[X]$ or is of *anti-integral type* if $H = I_H\varphi_H(X)R[X]$.

(2) The ideal H is called a *super-primitive ideal* of $R[X]$ or is of *super-primitive type* if $\mathrm{grade}(I_H\mathsf{c}(\varphi_H(X))) > 1$.

Remark 10.1.2 Assume that the ideal H of $R[X]$ is exclusive. Then $H_K \cap R[X] = H$. In particular, the inclusion, $I_H\varphi_H(X)R[X] \subseteq H$ holds. Indeed since $H_K \supseteq H$ asserts the inclusion, $H \subseteq H_K \cap R[X]$. Conversely, take any $f(X) \in \varphi_H K[X] \cap R[X]$. Then $f(X) = \varphi_H(X)\zeta(X)$ for some $\zeta(X) \in K[X]$. Since $f(X) \in H_K$, there exists a nonzero $a \in R$ such that $af(X) \in H$. Since H is exclusive over R, we have that $a \notin P$ for each $P \in Ass_{R[X]}(R[X]/H)$. Thus $f(X) \in H$, which implies that $H \supseteq H_K \cap R[X]$. $\square$

Proposition 10.1.3 *Assume that H is an exclusive ideal of $R[X]$. If H is generated by some polynomials of the least degree, then H is of anti-integral type.*

Proof Let d denote the least degree of a polynomial in H. Then $\varphi_H(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ ($\eta_i \in K$). Since $a\varphi_H(X) \in R[X]$ for some $a \in R \setminus \{0\}$. Take $f(X) = a_0X^d + a_1X^{d-1} + \cdots + a_d \in H$ with $a_i \in R$. Since $f(X) \in H$, we have $a_i/a_0 = \eta_i$ ($1 \le i \le d$). So $a_0 \in I_H$ and hence $f(X) = a_0\varphi_H(X)$. Thus $H \subseteq I_H\varphi_H(X)R[X]$. Since $I_H\varphi_H(X)R[X] \subseteq H$ by Remark 10.1.2, we have $H = I_H\varphi_H(X)R[X]$, which means that H is an anti-integral ideal. $\square$

Let $f(X) = a_0X^n + a_1X^{n-1} + \cdots + a_n$ be a polynomial in $R[X]$. We say that $f(X)$ is a *Sharma polynomial* in $R[X]$ if there does not exist $t \in R$ with $t \notin a_0R$ such that $ta_i \in a_0R$ for $1 \le i \le n$.

We require the following two lemmas seen in Chapter 3 (Theorem 2.1.2 and Theorem 2.1.8):

Lemma 10.1.4 *Let $f(X) \in R[X]$. Then $f(X)$ is a Sharma polynomial if and only if $\mathsf{c}(f(X)) \not\subseteq P$ for any $P \in \mathrm{Dp}_1(R)$.*

Lemma 10.1.5 *Assume that an ideal H is exclusive and that $\deg \varphi_H(X) = d$. The following statements are equivalent:*

(i) *H is a principle ideal of $R[X]$;*

(ii) *I_H is a principal ideal of R;*

(iii) *there exists a Sharma polynomial in H of degree d.*

If one of the above conditions holds, then H is generated by a Sharma polynomial.

Proof (iii) $\Rightarrow$ (i): Let $f(X)$ be a Sharma polynomial in H of degree d. Since $\deg \varphi_H(X) = d$, this Sharma polynomial has the least degree. So by [Sh], H is principal.
(i) $\Rightarrow$ (ii): Let $H = f(X)R[X]$. Then $f(X)R[X] \supseteq I_H\varphi_H(X)R[X]$. Note that $H \otimes_R K = f(X)K[X] = \varphi_H(X)K[X]$ and hence $\deg f(X) = \deg \varphi_H(X) = d$. Take $a \in I_H$. Then $a\varphi_H(X) = bf(X)$ for some $b \in R$. Let $f(X) = a_0X^d + \cdots + a_d$ with $a_i \in R$. Then $a = ba_0$, so that $I_H \supseteq a_0R$ for some $b \in R$. Since $ba_0\eta_i = a\eta_i = ba_i$ $(1 \le i \le d)$, we have $a_0\eta_i = a_i \in R$. Hence $a_0 \in I_H$, which implies that $I_H = a_0R$.
(ii) $\Rightarrow$ (iii): Let $I_H = bR$. Then $I_H\varphi_H(X)R[X] = b\varphi_H(X)R[X] \subseteq H$ and $b\eta_i \in R$ $(1 \le i \le d)$. Suppose that there exists $t \notin bR$ with $tb\eta_i \in bR$ $(1 \le i \le d)$. Then $t\eta_i \in R$ and hence $t \in I_H = bR$, a contradiction. Thus $b\varphi_H(X) \in R[X]$ is a Sharma polynomial of degree d. ∎

Proposition 10.1.6 *Assume that H is an exclusive ideal. If H is a super-primitive ideal of $R[X]$, then H is an anti-integral ideal of $R[X]$.*

Proof Since H is exclusive, $H \supseteq I_H\varphi_H(X)R[X]$ by Remark 10.1.2. Take $f(X) \in H$. Since $f(X) \in H_K$, there exists a nonzero element $a \in R$ such that $af(X) \in I_H\varphi_H(X)R[X]$. Take an arbitrary prime ideal $p \in \mathrm{Dp}_1(R)$. Then $I_H\mathsf{c}(\varphi_H(X)) \not\subseteq p$ because $\mathrm{grade}(I_H\varphi_H(X)R[X]) > 1$. So there exists a Sharma polynomial in H_p of the least degree. By Lemma 10.1.4, we have $H_p = g(X)R_p[X]$ for some $g(X) = b\varphi_H(X)$ and some $b \in R$. Since $f(X) \in H_p$, there exists $c \in R \setminus p$ such that $cf(X) \in I_H\varphi_H(X)R[X]$. Thus $f(X) \in I_H\varphi_H(X)R[X]$. ∎

Remark 10.1.7 If H is a super-primitive ideal of $R[X]$, then $\mathrm{grade}(\mathsf{c}(H)) > 1$. But the converse statement is not necessarily valid as can be seen in the example below. $\square$

Example 10.1.8 Consider an integral domain R satisfying the condition: $R \underset{\neq}{\subset} \overline{R}$, where $\overline{R}$ denotes the integral closure of R in K. Take $\alpha \in \overline{R}$ but $\alpha \notin R$. Then we have an exact sequence:

$$0 \to H \to R[X] \to R[\alpha] \to 0$$

Note that H is a prime ideal of $R[X]$ because $R[\alpha] \cong R[X]/H$ is an integral domain. In this case, H contains a monic polynomial (i.e., a polynomial giving an integral dependence of α). Hence $\mathsf{c}(H) = R$. Suppose that H is a super-primitive ideal, then H is an anti-integral ideal by Proposition 10.1.6. Since $H = I_{[\alpha]}\varphi_\alpha(X)R[X]$, we have $J_{[\alpha]} = I_{[\alpha]}\mathsf{c}(\varphi_\alpha(X)) = \mathsf{c}(H) = R$. Thus $R[\alpha]$ is flat over R (*cf.* Theorem 2.3.6). Since α is integral and flat in K, $R[\alpha] = R$, that is, $\alpha \in R$, which is a contradiction. Therefore H is not a super-primitive ideal.

Theorem 10.1.9 *Assume that H is an exclusive ideal of $R[X]$.*

(1) *If H is of anti-integral type and* $\mathrm{grade}(\mathsf{c}(H)) > 1$, *then H is of super-primitive type.*

(2) *If H is of anti-integral type and H contains a Sharma polynomial, then H is of super-primitive type.*

Proof Note that $H = I_H\varphi_H(X)R[X]$ by the assumption. Since $\mathsf{c}(H) = I_H\mathsf{c}(\varphi_H(X))$, (1) is valid. The statement (2) follows from Lemma 10.1.4. $\square$

Proposition 10.1.10 *Let $f(X)$ be a Sharma polynomial. Then $f(X)R[X]$ does not have any embedded prime divisor.*

Proof Let P be a prime divisor of $f(X)R[X]$. Then $\mathrm{depth}(R[X]_P) = 1$. Suppose that $p := P \cap R \neq (0)$. Then $\mathrm{depth}(R_p) = 1$ and $P = pR[X]$. Thus $f(X) \in P = pR[X]$, which implies that $\mathsf{c}(f(X)) \subseteq p$. Hence $\mathrm{grade}(\mathsf{c}(f(X))) = 1$. This contradicts the assumption that $f(X)$ is a Sharma polynomial. So we have $P \cap R = (0)$. Since $P = P_K \cap R[X]$ and $\mathrm{ht}(P) = 1$, $f(X)R[X]$ has no embedded prime divisor. $\square$

Remark 10.1.11 Let $f(X)$ be a Sharma polynomial in $R[X]$ and P be a prime divisor of $f(X)R[X]$. It does not necessarily follow that P is of

super-primitive type. But if P is of anti-integral type, then P is of super-primitive type. Indeed, consider the exact sequence:

$$0 \to P \to R[X] \to R[\alpha] \to 0$$

where α denotes $X \bmod. P$. In this case, since P contains a Sharma polynomial $f(X)$, α is a super-primitive element over R by Theorem 2.2.8. Next consider the case $R = \overline{R}$. If $P \cap R = (0)$, then P is of super-primitive type. ∎

Let H be an exclusive ideal of $R[X]$. Define $J_H := I_H \mathsf{c}(\varphi_H(X))$, an ideal of R.

Theorem 10.1.12 *Let H be an exclusive ideal of $R[X]$.*

(1) *If $J_H = R$, then $R[X]/H$ is flat over R.*
(2) *If $J_H = R$, then H is of super-primitive type and I_H is an invertible ideal of R.*

Proof (1) Take $p \in \operatorname{Spec}(R)$. Since $R = J_H = I_H \mathsf{c}(\varphi_H(X))$, there exists $b \in I_H$ such that $(I_H)_p$ is a principal ideal bR_p. So $(I_H)_p = b\varphi_H(X)R_p[X]$. It follows that $R_p[X]/(I_H)_p = R_p[X]/b\varphi_H(X)R_p[X]$. Thus $R_p[X]/b\varphi_H(X)$ $R_p[X]$ is flat over R_p by [M1, (20.F)] because $R_p = (J_H)_p = \mathsf{c}(b\varphi_H(X))_p$. Hence $R[X]/H$ is flat over R.
(2) Since $R = J_H = I_H \mathsf{c}(\varphi_H(X))$, I_H is an invertible ideal of R and H is a super-primitive ideal by definition. ∎

Corollary 10.1.13 *Let H be an exclusive ideal of $R[X]$. If $J_H = R$, then $H = I_H \varphi_H(X)R[X]$ and H is an invertible ideal of $R[X]$. Furthermore H^ℓ $(\ell > 0)$ is also an invertible ideal of $R[X]$.*

Proof By Theorem 10.1.12(2), H is of super-primitive type and hence H is of anti-integral type by Proposition 10.1.6. So $H = I_H \varphi_H(X)R[X]$, which is an invertible ideal of $R[X]$. ∎

Remark 10.1.14 Assume that H satisfies the condition in Corollary 10.1.13. Let $J := \{a \in R \mid a\varphi_H(X) \in R[X]\}$, an ideal of R. Then $H^\ell = J\varphi_H(X)^\ell R[X]$ and $J = (I_H)^\ell$. ∎

Proposition 10.1.15 *Let $f(X) \in R[X]$ be a Sharma polynomial. Assume that $f(X)$ is irreducible in $K[X]$. Then $f(X)R[X]$ is a prime ideal in $R[X]$.*

Proof Let P be a prime divisor of $f(X)R[X]$. Then $f(X) \in \varphi_P(X)K[X]$. Since $f(X)$ is irreducible, there exists $a \in R$ such that $f(X) = a\varphi_P(X)\,(a \in P)$.

Since $\mathrm{grade}(\mathsf{c}(f(X))) > 1$, P is of super-primitive type. So P is of anti-integral type by Proposition 10.1.6 and hence $P = I_P\varphi_P(X)R[X]$. Thus $f(X)R[X] = P$. $\square$

Proposition 10.1.16 *Let $f(X)$ be a Sharma polynomial in $R[X]$. Let P be a prime ideal in $R[X]$ such that P is a unique prime divisor of the ideal $f(X)R[X]$. Then P is of super-primitive type.*

Proof By the assumption, $f(X)K[X] = \varphi_P(X)^\ell K[X]$. Since $f(X)$ is a Sharma polynomial, we have $f(X)K[X] \cap R[X] = f(X)R[X]$ (*cf.* [Sh]). Hence $f(X)R[X] \subseteq (I_P\varphi_P(X))^\ell R[X]$. So, since $\mathrm{grade}(\mathsf{c}(f(X))) > 1$, we have $\mathrm{grade}(I_P\varphi_P(X)) > 1$. Hence P is of super-primitive type. $\square$

Lemma 10.1.17 *Let $\varphi(X) = X^d + \eta_1 X^{d-1} + \cdots \eta_d$ ($\eta_i \in K$) be a monic irreducible polynomial in $K[X]$. Put $I^R_{\varphi(X)} := \bigcap_{i=1}^d I_{\eta_i} (= R[X] :_R \varphi(X))$. If it holds that* $\mathrm{grade}(I^R_{\varphi(X)}\mathsf{c}(\varphi(X))) > 1$, *then $\varphi(X)K[X] \cap R[X] = I^R_{\varphi(X)}\varphi(X)R[X] \in$* $\mathrm{Spec}(R[X])$.

Proof Since $\varphi(X)K[X] \in \mathrm{Spec}(K[X])$, we have $P := \varphi(X)K[X] \cap R[X] \in \mathrm{Spec}(R[X])$. By construction, $P \supseteq I^R_{\varphi(X)}R[X]$. By the assumption,

$$\mathrm{grade}(I^R_{\varphi(X)}\mathsf{c}(\varphi(X))) > 1$$

Thus P is of super-primitive type. So we obtain $P = I^R_{\varphi(X)}\varphi(X)R[X] \in \mathrm{Spec}(R[X])$. $\square$

Proposition 10.1.18 *Let $f(X)$ be a Sharma polynomial in $R[X]$ and let $f(X) = a\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}$ be a product of irreducible polynomials $\varphi_i(X) \in K[X]$ and $a \in R$. Then $f(X)R[X] = (\varphi_1(X)^{e_1}K[X] \cap R[X]) \cap \cdots \cap (\varphi_t(X)^{e_t}K[X] \cap R[X])$ is a primary decomposition.*

Proof Since $f(X)$ is a Sharma polynomial, we have

$$\begin{aligned} f(X)R[X] &= f(X)K[X] \cap R[X] \\ &= \varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}K[X] \cap R[X] \\ &= (\varphi_1(X)^{e_1}K[X] \cap \cdots \cap (\varphi_t(X)^{e_t}K[X]) \cap R[X] \\ &= (\varphi_1(X)^{e_1}K[X] \cap R[X]) \cap \cdots \cap (\varphi_t(X)^{e_t}K[X] \cap R[X]) \end{aligned}$$

Since $f(X) \in \varphi_i(X)K[X] \cap R[X]$, the prime ideal $\varphi_i(X)K[X] \cap R[X]$ is of super-primitive type. So $\varphi_i(X)K[X] \cap R[X] = I^R_{\varphi_i(X)}R[X]$ by Lemma 10.1.17. $\varphi_i(X)^{e_i}K[X] \cap R[X]$ is an $I^R_{\varphi_i(X)}R[X]$-primary ideal. $\square$

Proposition 10.1.19 *Let $I \subseteq J$ be ideals of $R[X]$. Assume that $I \otimes_R K = J \otimes_R K$ and that, for each $p \in \mathrm{Dp}_1(R)$, JR_p contains a Sharma polynomial over R_p of the least degree. Then $I = J$.*

Proof Take $f(X) \in I$. Then $f(X) \in I \subseteq I \otimes_R K = J \otimes_R K$. Hence there exists $a \in R$ such that $af(X) \in J$. Let p be a prime divisor of aR. Then $p \in \mathrm{Dp}_1(R)$. By the assumption, $J_p = g(X)R_p[X]$, $\mathsf{c}(g(X)) = R_p$ for some $g(X) \in J$. Put $f(X) = g(X)\zeta(X)$ with $\zeta(X) \in R_p[X]$. Then there exists $b \in R \setminus p$ such that $bf(X) \in J$. Since a, b is a regular sequence, we have $f(X) \in J$. $\square$

10.2 Super-Primitive Polynomials and Sharma Polynomials

Definition 10.2.1 Let $\varphi(X)$ be a monic polynomial in $K[X]$ and $I^R_{\varphi(X)} := R[X] :_R \varphi(X)$. The polynomial $\varphi(X)$ is called a *super-primitive polynomial* if $\mathrm{grade}(I^R_\varphi \mathsf{c}(\varphi(X))) > 1$.

Let $\overline{R}$ denote the integral closure of R in K and $\mathcal{C}(\overline{R}/R)$ denote the conductor ideal between R and $\overline{R}$. For an element $\eta \in K$, we put $I_\eta := \{a \in R \mid a\eta \in R\}$.

Remark 10.2.2 Assume that $R \subsetneq \overline{R}$.

(1) There exists a polynomial in $K[X]$, which is not super-primitive. In fact, take $\eta \in \overline{R} \setminus R$ and $\varphi(X) := X - \eta$. Then $\varphi(X)$ is not super-primitive (*cf.* Example 10.2.9).

(2) When $\eta \in K$ satisfies $\mathrm{grade}(I_\eta + \mathcal{C}(\overline{R}/R)) > 1$, the polynomial $X - \eta$ is super-primitive. More generally, let $\varphi(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ ($\eta_i \in K$) satisfy $\mathrm{grade}(I_{\eta_i} + \mathcal{C}(\overline{R}/R)) > 1$ for all i. Then $\mathrm{grade}(\bigcap_{i=1}^d I_{\eta_i} + \mathcal{C}(\overline{R}/R)) = \mathrm{grade}(I^R_{\varphi(X)} + \mathcal{C}(\overline{R}/R)) > 1$. Hence $\varphi(X)$ is super-primitive (*cf.* Proposition 10.2.10). $\square$

Proposition 10.2.3 *Let B be an ideal of R. Assume that every prime divisor of B is contained in $\mathrm{Dp}_1(R)$. Let a, b be a regular sequence and $f \in R$. If $af, bf \in B$, then $f \in B$.*

Proof Consider a primary decomposition of B:

$$B = q_1 \cap \cdots \cap q_n$$

where $\sqrt{q_i} = p_i$. Then $p_i \in \mathrm{Dp}_1(R)$ for all $1 \leq i \leq n$. Take p_i. Then a, b is an R-regular sequence. So $a \notin p_i$ or $b \notin p_i$. If $a \notin p_i$, then $f \in q_i R_{p_i} \cap R = q_i$. A similar argument is applicable to the case $b \notin p_i$. Hence we conclude that $f \in B$. ∎

Proposition 10.2.4 *Let $\varphi(X) \in K[x]$ be an irreducible super-primitive polynomial. Let $I := I^R_{\varphi(X)}$ and $J := I^R_{\varphi(X)^\ell}$ ($\ell > 0$). Then $I\varphi(X)R[X]$ is a prime ideal of $R[X]$ and $J\varphi(X)^\ell R[X]$ is $I\varphi(X)R[X]$-primary ideal.*

Proof Since $\varphi(X)^\ell K[X]$ is $\varphi(X)K[X]$-primary, the ideal $\varphi(X)^\ell K[X] \cap R[X]$ is $\varphi(X)K[X] \cap R[X]$-primary. $\varphi(X)$ is super-primitive, and hence anti-integral. So $\varphi(X)K[X] \cap R[X] = I\varphi(X)R[X]$. Thus we have only to prove that $\varphi(X)^\ell K[X] \cap R[X] = J\varphi(X)^\ell R[X]$. The implication ($\supseteq$) is obvious. We shall show that the implication ($\subseteq$). Since J is a denominator ideal in R, J is a divisorial ideal. So $JR[X](\subseteq R[X])$ is also a divisorial ideal. Hence $J\varphi(X)^\ell R[X](\subseteq R[X]$ is a divisorial ideal. Thus any prime divisor of $J\varphi(X)^\ell R[X]$ is of depth 1. Take $f(X) \in \varphi(X)^\ell K[X] \cap R[X]$. Then $f(X)/\varphi(X)^\ell \in K[X]$. We must show that $f(X)/\varphi(X)^\ell \in JR[X]$. Let p be any prime divisor of J. Then $p \in \mathrm{Dp}_1(R)$. In this case, we have to show $f(X)/\varphi(X)^\ell \in (JR[X])_p$. Since $(I\varphi(X))^\ell R[X] \subseteq J\varphi(X)^\ell R[X]$, we have $\mathrm{grade}(J\mathrm{c}(\varphi(X)^\ell)) > 1$. Hence $J_p = aR_p$ for some $a \in J$. Put $g(X) = a\varphi(X)^\ell$. Then $g(X) \in J$ and $\mathrm{c}(g(X))R_p = R_p$, which shows $f(X) \in g(X)R_p[X]$. Thus $f(X)/\varphi(X)^\ell \in (g(X/\varphi(X)^\ell)R_p[X] = aR_p[X] = (JR[X])_p$. From this, we get $\varphi(X)^\ell K[X] \cap R[X] = J\varphi^\ell R[X]$. By Lemma 10.2.17, $I\varphi(X)R[X] \in \mathrm{Spec}(R[X])$. Therefore we have $J\varphi(X)^\ell R[X]$ is an $I\varphi(X)R[X]$-primary ideal. ∎

Theorem 10.2.5 *Let $\varphi_1(X), \ldots, \varphi_t(X) \in K[X]$ be irreducible super-primitive polynomials. Let $I_i := I^R_{\varphi_i(X)}$ ($1 \leq i \leq t$) and let $J_i := I^R_{\varphi_i(X)^{e_i}}$ ($1 \leq i \leq t, e_i > 0$). Then*

$$\varphi_1(X)^{e_1} \cdots \varphi_t(X)^{e_t} K[X] \cap R[X] = \bigcap_{i=1}^{t} (J_i \varphi_i(X)^{e_i} R[X])$$

and this is a primary decomposition of the left side ideal, where $J_i\varphi_i(X)^{e_i}R[X]$ is an $I_i\varphi_i(X)R[X]$-primary ideal for each $1 \leq i \leq t$.

Proof It follows that

$$\begin{aligned}
& \varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}K[X]\cap R[X] \\
= \ & (\varphi_1(X)^{e_1}K[X]\cap\cdots\cap\varphi_t(X)^{e_t}K[X])\cap R[X] \\
= \ & (\varphi_1(X)^{e_1}K[X]\cap R[X])\cap\cdots\cap(\varphi_t(X)^{e_t}K[X]\cap R[X]\cap R[X]) \\
= \ & J_1\varphi_1(X)^{e_1}R[X]\cap\cdots\cap J_t\varphi(X)^{e_t}R[X]
\end{aligned}$$

by Lemma 10.2.17. By Proposition 10.2.4, $J_i\varphi_i(X)^{e_i}R[X]$ is an $I_i\varphi_i(X)R[X]$-primary ideal for each $1\le i\le t$. ∎

Proposition 10.2.6 *Let $\varphi_1(X),\ldots,\varphi_t(X)\in K[X]$ be irreducible super-primitive polynomials. Then $\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}K[X]\cap R[X]_p$ is a principal ideal at each $p\in \mathrm{Dp}_1(R)$.*

Proof Let $J := R[X] :_R \varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}$ and $I_i := R[X] :_R \varphi_i(X)$. Since $\varphi_i(X)$ is a super-primitive polynomial, we have $\mathrm{grade}(I_i\mathsf{c}(\varphi_i(X)))>1$ for each $1\le i\le t$. Since $(I_1)^{e_1}\cdots(I_t)^{e_t}\subseteq J$, we have $\mathrm{grade}(J\mathsf{c}(\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t})>1$. Take $p\in\mathrm{Dp}_1(R)$. There exists $f_i(X)\in I_i\varphi_i(X)$ such that $\mathsf{c}(f_i(X))\not\subseteq p$ for $1\le i\le t$. Put $f(X) := f_1(X)^{e_1}\cdots f_t(X)^{e_t}$. Then $\mathsf{c}(f_i(X))\not\subseteq p$. Since $f(X)\in\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}K[X]\cap R[X]$ and $f(X)$ has the least degree, we have

$$(\varphi_1(X)^{e_1}\cdots\varphi_t(X)^{e_t}K[X]\cap R[X])_p = f(X)R[X]_p$$

by Corollary 2.1.6. ∎

Remark 10.2.7 Let $\varphi(X)\in K[X]$ be a monic polynomial and $I := I^R_{\varphi(X)}$. If $I\varphi(X)R[X]$ contains a Sharma polynomial, then $\varphi(X)$ is a super-primitive polynomial. Indeed, take a Sharma polynomial $f(X)\in I\varphi(X)R[X]$. Suppose there exists $p\in\mathrm{Dp}_1(R)$ such that $I\mathsf{c}(\varphi(X))\subseteq p$. Then $I\varphi(X)R[X]\subseteq pR[X]$ and hence $f(X)\in pR[X]$. Thus $\mathsf{c}(f(X))\subseteq p$, which contradicts the assumption that $f(X)$ is a Sharma polynomial. ∎

Theorem 10.2.8 *Let $\varphi(X)\in K[X]$ be a monic polynomial. Then $\varphi(X)$ is a super-primitive polynomial if and only if $I^R_{\varphi(X)}\varphi(X)R[X]$ contains a Sharma polynomial.*

Proof The implication ($\Leftarrow$) follows from Remark 10.2.7.
($\Rightarrow$) Take a nonzero element $a\in I^R_{\varphi(X)}$ and put $f_1(X)=a\varphi(X)$. Let $p_1,\ldots,p_n\in\mathrm{Dp}_1(R)$ such that $p_i\supseteq\mathsf{c}(f_1(X))$. Since $\varphi(X)$ is a super-primitive polynomial, we have $I^R_{\varphi(X)}\mathsf{c}(\varphi(X))\not\subseteq p_1,\ldots,p_n$. There exists $g_i(X)\in I^R_{\varphi(X)}\varphi(X)R[X]$

such that $\mathsf{c}(g_i(X)) \not\subseteq p_i$ $(1 \leq i \leq n)$. Put $f(X) = f_1(X)g_i(X)\cdots g_n(X)$. In this case, $\mathsf{c}(f(X)) \subseteq \mathsf{c}(f_1(X)), \mathsf{c}(g_i(X))$ $(1 \leq i \leq n)$. Thus for each $p \in \mathrm{Dp}_1(R)$, $\mathsf{c}(f(X)) \not\subseteq p$. Hence $f(X)$ is a Sharma polynomial and $f(X) \in I^R_{\varphi(X)}\varphi(X)R[X]$. $\square$

The following example shows that an irreducible factor $\varphi(X)$ in $K[X]$ of a Sharma polynomial $f(X)$ is not always a super-primitive polynomial.

Example 10.2.9 Assume that $R \underset{\neq}{\subset} \overline{R}$. Take $\alpha \in \overline{R} \setminus R$. Since α is integral over R, there exists a monic polynomial in $R[X]$ such that $f(\alpha) = 0$. Note that $\mathsf{c}(f(X)) = R$ because $f(X)$ is a monic. Thus $f(X)$ is a Sharma polynomial. It is obvious that $X - \alpha$ is a factor of $f(X)$ in $K[X]$. Suppose $X - \alpha$ is anti-integral. Then $f(X) \in (X - \alpha)K[X] \cap R[X] = I_\alpha(X - \alpha)R[X]$. So $R = \mathsf{c}(f(X)) \subseteq I_\alpha \mathsf{c}(X - \alpha)$, which means that $R[\alpha] \cong R[X]/(X - \alpha)$ is flat over R (*cf.* Theorem 2.3.6). Since α is integral over R, we have $R[\alpha] = R$, that is, $\alpha \in R$, which is a contradiction. Therefore the polynomial $X - \alpha$ is not an anti-integral polynomial, and hence $X - \alpha$ is not a super-primitive polynomial.

Proposition 10.2.10 *Let $\varphi(X)$ be a monic polynomial in $K[X]$.*

(1) *If* $\mathrm{grade}(I^R_{\varphi(X)} + \mathcal{C}(\overline{R}/R)) > 1$, *then* $\mathrm{grade}(I^R_{\varphi(X)}\mathsf{c}(\varphi(X))) > 1$.

(2) $\mathrm{grade}(I^R_{\varphi(X)}\mathsf{c}(\varphi(X))) > 1$ *if and only if* $I^R_{\varphi(X)}\varphi(X)R[X]$ *contains a Sharma polynomial.*

(3) *If* $R = \overline{R}$, *then* $\mathrm{grade}(I^R_{\varphi(X)}\mathsf{c}(\varphi(X))) > 1$.

Proof (1) Take $p \in \mathrm{Dp}_1(R)$. If $I^R_{\varphi(X)} \not\subseteq p$, then $I^R_{\varphi(X)} \subseteq I^R_{\varphi(X)}\mathsf{c}(\varphi(X))$. So $I^R_{\varphi(X)} \not\subseteq p$. On the other hand, if $I^R_{\varphi(X)} \subseteq p$, then $\mathcal{C}(\overline{R}/R) \not\subseteq p$ by the assumption and hence R_p is a DVR. Let $v_p(\ )$ denote the valuation of R_p. If $v_p(I^R_{\varphi(X)}) = e$, then $(I^R_{\varphi(X)})_p = p^e R_p$ and there exists i such that $V_p(\eta_i) = -e > 0$, where $\varphi(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ with $\eta_j \in K$. Thus $v_p(I^R_{\varphi(X)}\eta_i) = 0$. Hence $(I^R_{\varphi(X)}\eta_i)_p \not\subseteq pR_p$. Thus $(I^R_{\varphi(X)}\mathsf{c}(\varphi(X)))_p \not\subseteq pR_p$. This shows that $I^R_{\varphi(X)}\mathsf{c}(\varphi(X)) \not\subseteq p$ Therefore we obtain that $\mathrm{grade}(I^R_{\varphi(X)}\mathsf{c}(\varphi(X))) > 1$.
(2)($\Rightarrow$): Take $a \in I^R_{\varphi(X)}$ and put $f(X) = a\varphi(X)$. If $f(X)$ is a Sharma polynomial, then there is nothing to prove. Suppose that $f(X)$ is not a Sharma polynomial. Then there exists a prime ideal p in $\mathrm{Dp}_1(R)$ such that $\mathsf{c}(f(X)) \subseteq p$. Take a nonzero element $b \in \mathsf{c}(f(X))$. Then p is a prime divisor of the ideal bR because $p \in \mathrm{Dp}_1(R)$. Since the number of prime divisors of bR is finite, the number of prime ideals in $\mathrm{Dp}_1(R)$ containing $\mathsf{c}(f(X))$ is also finite. Let $p_1, \ldots, p_n$ be the prime ideals containing $\mathsf{c}(f(X))$. Since $I^R_{\varphi(X)}\mathsf{c}(\varphi(X)) \not\subseteq p_i$ for all $1 \leq i \leq n$

by the assumption, there exists $g_i(X) \in I^R_{\varphi(X)}\varphi(X)$ such that $g_i(X) \notin p_i R[X]$ for each $1 \le i \le n$. Put $g(X) := f(X) + X^d g_1(X) + \cdots + X^{nd} g_n(X)$. Then $\mathsf{c}(g(X)) = \sum_{i=1}^n (\mathsf{c}(g_i(x)) + \mathsf{c}(f(X)) \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R)$. Thus $g(X)$ is a Sharma polynomial.
(2)($\Leftarrow$): Let $f(X)$ be a Sharma polynomial in $I^R_{\varphi(X)}\varphi(X)R[X]$. Then we can express $f(X) = \sum f_i(X)g_i(X)$ with $f(X) \in I^R_{\varphi(X)}\varphi(X)$ and $g_i(X) \in R[X]$. Suppose that there exists $p \in \mathrm{Dp}_1(R)$ such that $I^R_{\varphi(X)}\mathsf{c}(\varphi(X)) \subseteq p$. Then $f_i(X) \in pR[X]$ and hence $f(X) \in pR[X]$, which contradicts the assumption that $f(X)$ is a Sharma polynomial.
(3) follows from $\mathcal{C}(\overline{R}/R) = R$ and (1). ∎

In Proposition 10.2.10, the converse implication of (1) is not always valid as can be seen in the following example.

Example 10.2.11 Let k be a field and t be an indeterminate. Let $R := k[t^2, t^3]$. Then $\overline{R} = k[t]$. Let $\varphi(X) := X^2 + X + 1/t^2 \in K[X] := k(t)[X]$. Then $I^R_{\varphi(X)} = t^2 R$ and $I^R_{\varphi(X)}\mathsf{c}(\varphi(X)) = t^2(1, 1/t^2)R = R$. So we have $\mathrm{grade}(I^R_{\varphi(X)}\mathsf{c}(\varphi(X))) = \mathrm{grade}(R) > 1$, but $\mathrm{grade}(I^R_{\varphi(X)} + \mathcal{C}(\overline{R}/R)) = \mathrm{grade}(t^2R + tR) = \mathrm{grade}(tR) = 1$.

Remark 10.2.12 Consider a Noetherian domain R satisfying $R \subsetneq \overline{R}$ and let $\varphi(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d \in K[X]$. Assume that every $\eta_i \in \overline{R}$ and that $\eta_j \notin R$ for some j. Then $I^R_{\varphi(X)}\varphi(X)R[X]$ does not contain any Sharma polynomial. ∎

10.3 Anti-Integral, Super-Primitive, or Flat Polynomials

Definition 10.3.1 Let $\varphi(X) \in K[X]$ be a monic polynomial. We say that $\varphi(X)$ is an *anti-integral polynomial* if $\varphi(X)K[X] \cap R[X] = I^R_{\varphi(X)}\varphi(X)R[X]$. We say that $\varphi_\alpha(X)$ is a *super-primitive polynomial* if $\varphi_\alpha(X)$ is anti-integral and

$$\mathrm{grade}(I^R_{\varphi(X)}\mathsf{c}(\varphi(X))) > 1$$

Proposition 10.3.2 *Let $\varphi(X) \in K[X]$ be a monic polynomial. If the ideal $I^R_{\varphi(X)}\varphi(X)R[X]$ contains a monic polynomial in $R[X]$, then $I^R_{\varphi(X)} = R$, i.e., $\varphi(X) \in R[X]$.*

Proof Let $f(X) \in I^R_{\varphi(X)}\varphi(X)R[X]$ be a monic polynomial in $R[X]$. Then $f(X) = \sum a_i\varphi(X)g_i(X) = (\sum a_i g_i(X))\varphi(X)$ with some $a_i \in I^R_{\varphi(X)}$ and some $g_i(X) \in R[X]$. Comparing the coefficients of the leading terms, we conclude that $I^R_{\varphi(X)} = R$. ∎

Corollary 10.3.3 *Let $\varphi(X) \in K[X]$ be a monic polynomial. Assume that $\varphi(X)$ is a factor of a monic polynomial in $R[X]$ and that $\varphi(X)$ is an anti-integral polynomial. Then $\varphi(X) \in R[X]$.*

Proof By the assumption, there exists a monic polynomial $f(X) \in \varphi(X)K[X] \cap R[X]$. So we have $f(X) \in \varphi(X)K[X] \cap R[X] = I_{\varphi(X)}\varphi(X)R[X]$. Hence $\varphi(X) \in R[X]$ by Proposition 10.3.2. ∎

The following example shows that every polynomial $\varphi(X) \in K[X]$ is not always an anti-integral polynomial.

Example 10.3.4 Assume that $R \subsetneq \overline{R}$ and take $\eta \in \overline{R} \setminus R$. Then $\varphi(X) := X - \eta$ is not an anti-integral polynomial. Indeed, Proposition 10.3.2 shows that $\varphi(X) = X - \eta$ is an anti-integral polynomial if and only if $\eta \in R$. So if $\varphi(X)$ is an anti-integral polynomial, then $\eta \in R$, which is a contradiction.

It is easy to see the following remarks by definition.

Remark 10.3.5 Let $\varphi(X) \in K[X]$ be a monic polynomial and $H := \varphi(X)K[X] \cap R[X]$, an ideal of $R[X]$.

(1) H is an exclusive ideal in $R[X]$.
(2) $\varphi(X) \in K[X]$ is an anti-integral polynomial if and only if H is an anti-integral ideal of $R[X]$.
(3) $\varphi(X) \in K[X]$ is a super-primitive polynomial if and only if H is a super-primitive ideal of $R[X]$. ∎

Remark 10.3.6 When H is a prime ideal, H is an anti-integral (resp. a super-primitive) ideal if and only if the extension $R[\alpha]$ of R is an anti-integral (resp. a super-primitive) extension of R as in Chapter 2, where α denotes the residue class of X in $R[X]/H$. ∎

Proposition 10.3.7 *Let $\varphi(X) \in K[X]$ be a monic polynomial. If $\varphi(X)$ is a super-primitive polynomial, then $\varphi(X)$ is an anti-integral polynomial.*

Proof Since $\varphi(X)$ is super-primitive, $H := \varphi(X)K[X] \cap R[X]$ is a super-primitive ideal of $R[X]$. Note that the ideal H is exclusive. Hence the conclusion follows from Proposition 10.1.6. $\square$

Lemma 10.3.8 *If both $f(X)$ and $g(X) \in R[X]$ are Sharma polynomials, then $f(X)g(X)$ is a Sharma polynomial.*

Proof Suppose that there exists $p \in \mathrm{Dp}_1(R)$ such that $f(X)g(X) \in pR[X]$. Then $f(X) \in pR[X]$ or $g(x) \in pR[X]$ and hence $\mathsf{c}(f(X)) \subseteq p$ or $\mathsf{c}(g(X)) \subseteq p$, which is a contradiction by Lemma 10.1.6. $\square$

Theorem 10.3.9 *Let $\varphi_1(X), \ldots, \varphi_n(X) \in K[X]$ be super-primitive polynomials. Then*

(1) $\varphi(X) := \varphi_1(X) \cdots \varphi_n(X) \in K[X]$ *is a super-primitive polynomial;*
(2) *if* $\varphi(X) \in R[X]$, $\varphi_i(X) \in R[X]$ *for each* $1 \leq i \leq n$.

Proof (1) Since $\varphi_i(X)$ is super-primitive, we have $I^R_{\varphi_i(X)}\mathsf{c}(\varphi_i(X)) \not\subseteq p$ for every $p \in \mathrm{Dp}_1(R)$. Take $p \in \mathrm{Dp}_1(R)$. Then there exists $a_i \in I^R_{\varphi_i(X)}$ such that $\mathsf{c}(a_i\varphi_i(X)) \not\subseteq p$. Thus $a_i\varphi_i(X)$ is a Sharma polynomial in $R_p[X]$. Put $a := a_1 \cdots a_n$. Then $a \in I^R_{\varphi(X)}$ and $a\varphi(X) = (a_1\varphi_1(X)) \cdots (a_n\varphi_n(X))$. So by Lemma 10.3.8, $a\varphi(X)$ is a Sharma polynomial in $R_p[X]$ and hence $\mathsf{c}(a\varphi(X)) \not\subseteq p$. Therefore we have $\mathrm{grade}(I^R_{\varphi(X)}\mathsf{c}(\varphi(X))) > 1$. So $a\varphi(X)$ is a super-primitive polynomial.
(2) follows from Proposition 10.3.7 and Corollary 10.3.3. $\square$

Question. Is the similar statement to Proposition 10.3.9 valid for anti-integral polynomials?

The following proposition asserts that a partial answer to the above question is valid.

Proposition 10.3.10 *Let $\varphi_1(X), \ldots, \varphi_n(X) \in K[X]$ be anti-integral polynomials. If $\varphi(X) := \varphi_1(X) \cdots \varphi_n(X) \in R[X]$ is a monic polynomial, then $\varphi_i(X) \in R[X]$ for all $1 \leq i \leq n$.*

Proof Since $\varphi_i(X)$ is an anti-integral polynomial, $\varphi(X) \in \varphi_i(X)K[X] \cap R[X] = I^R_{\varphi_i(X)}\varphi_i(X)R[X]$. So $I^R_{\varphi_i(X)} = R$ by Proposition 10.3.2. Thus we conclude that $\varphi_i(X) \in R[X]$. $\square$

Example 10.3.11 Let $R = k[t^2, t^3]$, where k is a field and t is an indeterminate. Let $\varphi_1(X) := X - t, \varphi_2(X) := X + t \in K[X]$. Then neither $\varphi_1(X)$ nor $\varphi_2(X)$ is an anti-integral polynomial. But $\varphi(X) := \varphi_1(X)\varphi_2(X) = (X - t)(X + t) = X^2 - t^2$ is an anti-integral polynomial.

Proposition 10.3.12 *Let $\varphi(X) \in K[X]$ and let $\varphi(X) = \varphi_1(X)^{e_1} \cdots \varphi_n(X)^{e_n}$ be the irreducible decomposition in $K[X]$. Assume that $\varphi_i(X)$ is a super-primitive polynomial for each $1 \leq i \leq n$. Then $I^R_{\varphi(X)} = (((I^R_{\varphi_1(X)})^{e_1} \cdots (I^R_{\varphi_n(X)})^{e_n})^{-1})^{-1}$.*

Proof $(I^R_{\varphi_1(X)})^{e_1} \cdots (I^R_{\varphi_n(X)})^{e_n} \subseteq I^R_{\varphi(X)}$ is obvious. Since $I^R_{\varphi(X)}$ is a divisorial ideal, we have $(((I^R_{\varphi_1(X)})^{e_1} \cdots (I^R_{\varphi_n(X)})^{e_n})^{-1})^{-1} \subseteq I^R_{\varphi(X)}$. Note that both sides of this implication are divisorial. Consider the localization at a prime ideal of depth 1. We may assume that (R, m) is a local domain with the maximal ideal $m \in \mathrm{Dp}_1(R)$. Since $\varphi_i(X)$ is a super-primitive polynomial, there exists $a_i \in I^R_{\varphi_i(X)}$ such that $a_i\varphi_i(X)$ is a Sharma polynomial by Lemma 10.1.4. Put $a := a_1 \cdots a_n$. Then $(a_1\varphi_1(X))^{e_1} \cdots (a_n\varphi_n(X))^{e_n} = a\varphi_1(X)^{e_1} \cdots \varphi_n(X)^{e_n}$ is a Sharma polynomial by Lemma 10.3.8. Thus $I^R_{\varphi(X)}\varphi(X)R[X] = a\varphi(X)R[X]$ and hence $I^R_{\varphi(X)} = aR$. Therefore $I^R_{\varphi(X)} = (((I^R_{\varphi_1(X)})^{e_1} \cdots (I^R_{\varphi_n(X)})^{e_n})^{-1})^{-1}$. ∎

Definition 10.3.13 Let $\varphi(X) \in K[X]$ be a monic polynomial. We say that $\varphi(X)$ is a *flat polynomial* if $I^R_{\varphi(X)}\mathsf{c}(\varphi(X)) = R$.

Theorem 10.3.14 *Let $\varphi(X)$, $\psi(X)$ be flat polynomials. Then*

(1) *$\varphi(X)\psi(X)$ is also a flat polynomial;*
(2) *$I^R_{\varphi(X)\psi(X)} = I^R_{\varphi(X)}I^R_{\psi(X)}$;*
(3) *$\varphi(X)$ is a super-primitive polynomial.*

Proof Note that

$$I^R_{\varphi(X)} \cdot I^R_{\psi(X)} \subseteq I^R_{\varphi(X)} \cap I^R_{\psi(X)} \subseteq I^R_{\varphi(X)\psi(X)}$$

Hence $I^R_{\varphi(X)}I^R_{\psi(X)}(\varphi(X)\psi(X))R[X] \subseteq I^R_{\varphi(X)\psi(X)}\varphi(X)\psi(X)R[X]$.
(1) Hence $R = I^R_{\varphi(X)}I^R_{\psi(X)}\mathsf{c}(\varphi(X)\psi(X)) \subseteq I^R_{\varphi(X)\psi(X)}\mathsf{c}(\varphi(X)\psi(X)) \subseteq R$. So $\varphi(X)\psi(X)$ is a flat polynomial.
(2) By the argument in the proof of (1), $\mathsf{c}(\varphi(X)\psi(X))$ is an invertible ideal of R. Thus $I^R_{\varphi(X)\psi(X)} = I^R_{\varphi(X)}I^R_{\psi(X)}$.
(3) follows from the definition. ∎

Proposition 10.3.15 *Let $\varphi(X) \in K[X]$ be a monic polynomial and let $\varphi(X) = \varphi_1(X)^{e_1} \cdots \varphi_n(X)^{e_n}$ be an irreducible decomposition in $K[X]$. If $\varphi_i(X)$ is a flat polynomial for each $1 \leq i \leq n$, then $I^R_{\varphi(X)} = (I^R_{\varphi_1(X)})^{e_1} \cdots (I^R_{\varphi_n(X)})^{e_n}$ and $I^R_{\varphi(X)}\varphi(X)R[X] = (\prod_{i=1}^n (I^R_{\varphi_i(X)})^{e_i} \varphi_i(X)^{e_i})R[X]$ is a primary decomposition.*

Proof This follows form Proposition 10.3.12 and Proposition 10.3.14 (1)(2). ∎

Definition 10.3.16 Let $\varphi(X) \in K[X]$ be a monic polynomial. We say that $\varphi(X)$ is an *ultra-primitive polynomial* if $\mathrm{grade}(I^R_{\varphi(X)} + \mathcal{C}(\overline{R}/R)) > 1$.

Theorem 10.3.17 *Let $\varphi(X), \psi(X) \in K[X]$ be monic polynomials. If both $\varphi(X)$ and $\psi(X)$ are ultra-primitive polynomials, then so is $\varphi(X)\psi(X)$.*

Proof Note that $I^R_{\varphi(X)} \cdot I^R_{\psi(X)} \subseteq I^R_{\varphi(X)} \cap I^R_{\psi(X)} \subseteq I^R_{\varphi(X)\psi(X)}$. Since $\mathrm{grade}(I^R_{\varphi(X)} + \mathcal{C}(\overline{R}/R)) > 1$ and $\mathrm{grade}(I^R_{\psi(X)} + \mathcal{C}(\overline{R}/R)) > 1$, we have $1 < \mathrm{grade}(I^R_{\varphi(X)} I^R_{\psi(X)} + \mathcal{C}(\overline{R}/R)) \leq \mathrm{grade}(I^R_{\varphi(X)\psi(X)} + \mathcal{C}(\overline{R}/R))$. ∎

Remark 10.3.18

(1) We have the following implications: an ultra-primitive polynomial $\Rightarrow$ a super-primitive polynomial $\Rightarrow$ an anti-integral polynomial.
The reverse implications are not valid in general.

(2) Take $\eta \in K$.
   (i) $X - \eta$ is a super-primitive polynomial if and only if $\mathrm{grade}(I_\eta(1, \eta)) > 1$.
   (ii) $X - \eta$ is an ultra-primitive polynomial if and only if $\mathrm{grade}(I_\eta + \mathcal{C}(\overline{R}/R)) > 1$.
   (iii) $X - \eta$ is a flat polynomial if and only if $I_\eta(1, \eta) = R$.

∎

Definition 10.3.19 Let $\eta \in K$.

1) η is called a *super-primitive element* if $\mathrm{grade}(I_\eta(1, \eta)) > 1$.
2) η is called an *ultra-primitive element* if $\mathrm{grade}(I_\eta + \mathcal{C}(\overline{R}/R)) > 1$.
3) η is called an *s-flat element* if and only if $I_\eta(1, \eta) = R$.

Remark 10.3.20 The above definition is the same as in Chapter 2, that is, $\eta \in K$ is super-primitive over R if the extension $R[\eta]$ is a super-primitive

extension of R as in Chapter 2. An element $\eta \in K$ is an s-flat element over R if and only if η is anti-integral over R and $R[\eta]$ is a flat extension of R. $\square$

Proposition 10.3.21 *Let $\varphi(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d \in K[X]$. If η_i is an ultra-primitive element for each $1 \leq i \leq d$, then $\varphi(X)$ is an ultra-primitive polynomial.*

Proof Note that $I_{\eta_1} \cdots I_{\eta_d} \subseteq \bigcap_{i=1}^{d} I_{\eta_i}$. Since $\operatorname{grade}(I_{\eta_i} + \mathcal{C}(\overline{R}/R)) > 1$ by definition, $\operatorname{grade}(I_{\eta_1} \cdots I_{\eta_d} + \mathcal{C}(\overline{R}/R)) > 1$ and hence $\operatorname{grade}(\bigcap_{i=1}^{d} I_{\eta_i} + \mathcal{C}(\overline{R}/R)) > 1$. Therefore $\operatorname{grade}(I^R_{\varphi(X)} + \mathcal{C}(\overline{R}/R)) > 1$ because $I^R_{\varphi(X)} = \bigcap_{i=1}^{d} I_{\eta_i}$, which implies that $\varphi(X)$ is an ultra-primitive polynomial. $\square$

In the notations as in Proposition 10.3.21, even if η_i is a super-primitive (respectively s-flat) element for each $1 \leq i \leq d$, $\varphi(X)$ is not necessarily a super-primitive (respectively s-flat) polynomial. We see this in the following example.

Example 10.3.22 Note first that an s-flat element (respectively an s-flat polynomial) is a super-primitive element (respectively a super-primitive polynomial). Let $R := k[t^2, t^3]$, where k is a field, and t be an indeterminate. Put $\eta_1 := 1/t^3$, $\eta_2 := 1/t^4$, and $\varphi(X) := X^2 + \eta_1 X + \eta_2 = X^2 + (1/t^3)X + (1/t^4)$. Then $I_{\eta_1} = t^3 R$, $I_{\eta_2} = t^4 R$. Hence $I_{\eta_1}(1, \eta_1) = I_{\eta_2}(1, \eta_2) = R$. Thus η_1 and η_2 are s-flat elements and hence they are super-primitive elements. We have $I^R_{\varphi(X)} = I_{\eta_1} \cap I_{\eta_2} = t^3 R \cap t^4 R = (t^6, t^7)R$. So we have $I^R_{\varphi(X)}\mathsf{c}(\varphi(X) = I^R_{\varphi(X)}(1, \eta_1, \eta_2) = (t^6, t^7)(1, 1/t^3, 1/t^4)R = (t^2, t^3)R \subsetneq R$. Thus $\operatorname{grade}(I^R_{\varphi(X)}\mathsf{c}(\varphi(X))) = 1$, which means that $\varphi(X)$ is not a super-primitive polynomial.

Chapter 11

Semi Anti-Integral and Pseudo-Simple Extensions

This chapter contains some supplementary results.

11.1 Anti-Integral Extensions of Polynomial Rings

Let R be a Noetherian domain. In Chapters 1 and 2, we investigated some properties of a simple algebraic extension $R[\alpha]$ in terms of ideals of R. Our objective of this section is to study about a simple birational extension $R[X][\alpha]$ of a polynomial ring $R[X]$ by means of ideals of R, and hence α is an element in $K(X)$. We observe mainly what conditions are to be required for α being super-primitive over $R[X]$ or $R[X][\alpha]$ being flat over $R[X]$.

In what follows, we fix the following notations unless otherwise specified: Let R denote a Noetherian domain, K denote its quotient field, and $R[X]$ denote a polynomial ring over R. Let $\eta(X)$ be an element in the quotient field $K(X)$ of $R[X]$ and put $\eta(X) = \psi(X)/\phi(X)$, where $\psi(X), \phi(X) \in K[X]$ with $(\phi(X), \psi(X))K[X] = K[X]$ and $\phi(X)$ monic. Note that expression $\eta(X)/\phi(X)$ is unique. Let $\deg \phi(X) =: \ell$.

We start with the following definition.

Definition 11.1.1 We say that $\eta(X)$ satisfies the *condition* (F) over R if the ideal $R[X] :_{R[X]} \eta(X) := \{f(X) \in R[X] \mid f(X)\eta(X) \in R[X]\}$ of $R[X]$ is generated by some polynomials of degree ℓ.

For $\zeta(X) \in K[X]$, put $I^R_{\zeta(X)} := \{a \in R \mid a\zeta(X) \in R[X]\}$, which is an ideal of R.

Proposition 11.1.2 *The element $\eta(X)$ satisfies the condition* (F) *if and only if* $R[X] :_{R[X]} \eta(X) = (I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X]$.

Proof ($\Leftarrow$) is trivial.
($\Rightarrow$) Since $\eta(X)$ satisfies the condition (F), $R[X] :_{R[X]} \eta(X)$ is generated by some polynomials of degree ℓ. Take $h(X) \in R[X] :_{R[X]} \eta(X)$ with $\deg h(X) = \ell$. Put $g(X) = h(X)\eta(X) \in R[X] \subseteq K[X]$. Then we have $h(X)\psi(X) \in \phi(X)K[X]$. Since $(\phi(X), \psi(X))K[X] = K[X]$, it follows that $h(X) \in \phi(X)K[X]$. So $\deg h(X) = \deg \phi(X) = \ell$ implies that $h(X) = \zeta\phi(X)$ with $\zeta \in K$. Since $\phi(X)$ is monic, the equality $\zeta\phi(X) = h(X) \in R[X]$ yields $\zeta \in I^R_{\phi(X)} \subseteq R$. Note that $g(X) = h(X)\eta(X) = \zeta\psi(X) \in R[X]$. Hence $\zeta \in I^R_{\psi(X)}$. Thus $\zeta \in I^R_{\phi(X)} \cap I^R_{\psi(X)}$. Therefore we have $h(X) \in (I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X]$. Conversely the implication $(I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X] \subseteq R[X] :_{R[X]} \eta(X)$ is obvious. ∎

For $f(X) = a_0X^n + a_1X^{n-1} + \cdots + a_n \in K[X]$, let $\mathsf{c}(f(X))$ denote the fractional ideal $(a_0, a_1, \ldots, a_n)R$ of $f(X)$ in K.

Lemma 11.1.3 *Let $f(X), g(X) \in R[X]$ which are co-prime in $K[X]$, i.e., $(f(X), g(X))K[X] = K[X]$. Then*

$$\mathrm{grade}((f(X), g(X))R[X]) > 1 \Leftrightarrow \mathrm{grade}((\mathsf{c}(f(X)), \mathsf{c}(g(X)))R) > 1$$

Proof ($\Rightarrow$) Suppose that $(\mathsf{c}(f(X)), \mathsf{c}(g(X)))R \subseteq p$ for some $p \in \mathrm{Dp}_1(R)$. Then $(f(X), g(X))R[X] \subseteq pR[X] = P \in \mathrm{Dp}_1(R[X])$, a contradiction.
($\Leftarrow$) Suppose that $(f(X), g(X))R[X] \subseteq P$ for some $P \in \mathrm{Dp}_1(R[X])$. When $P \cap R = (0)$, there exists an irreducible polynomial $\zeta(X) \in K[X]$ such that $P = \zeta(X)K[X] \cap R[X]$, which contradicts the assumption that $f(X), g(X)$ are co-prime in $K[X]$. Next consider the case $P \cap R \neq (0)$. Then $P \in \mathrm{Dp}_1(R[X])$ implies that $p := P \cap R \in \mathrm{Dp}_1(R)$ and $P = pR[X]$. In this case, we have $(\mathsf{c}(f(X)), \mathsf{c}(g(X)))R \subseteq p$, a contradiction. ∎

The following proposition shows that the condition (F) is determined by $\mathrm{Dp}_1(R)$.

Proposition 11.1.4 *$\eta(X)$ satisfies the condition* (F) *over R if and only if $\eta(X)$ satisfies the condition* (F) *over R_p for each $p \in \mathrm{Dp}_1(R)$.*

Proof By Proposition 11.1.2, $\eta(X)$ satisfies the condition (F) over R if and only if $R[X] :_{R[X]} \eta(X) = (I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X]$. Since $R[X] :_{R[X]} \eta(X)$ and $(I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X]$ are divisorial ideals of $R[X]$ (and then contractions in R are of depth ≤ 1), this equality holds if and only if it holds by localizing at each $p \in \mathrm{Dp}_1(R)$. ∎

Lemma 11.1.5 *Let $\phi(X) \in K[X]$ and let H be a divisorial ideal of R satisfying $H \subseteq I^R_{\phi(X)}$. If* $\mathrm{grade}(H\mathsf{c}(\phi(X))) > 1$, *then $\phi(X)K[X] \cap R[X] = H\phi(X)R[X]$.*

Proof Since $H \subseteq I^R_{\phi(X)}$, we have $\phi(X)K[X] \cap R[X] \supseteq H\phi(X)R[X]$. We shall show the converse inclusion. For this, we have only to prove $(1/\phi(X))(\phi(X)K[X] \cap R[X]) \subseteq HR[X]$. Since H is a divisorial ideal of R, $HR[X]$ is also a divisorial ideal of $R[X]$. So localizing at each $p \in \mathrm{Dp}_1(R)$, we need to prove the desired inclusion. Hence we may assume that (R, m) is a local ring of $\mathrm{depth}(R) = 1$. By the assumption, $H\mathsf{c}(\phi(X)) = R$. Thus H is an invertible ideal and hence a principal ideal $H = aR$. Put $f(X) = a\phi(X)$. Then $\mathsf{c}(f(X)) = R$. By [KY7], we have $\phi(X)K[X] \cap R[X] = f(X)R[X] = a\phi(X)R[X] = H\phi(X)R[X]$. ∎

Remark 11.1.6 It is shown that if $\mathrm{grade}(I^R_{\phi(X)}\mathsf{c}(\phi(X))) > 1$ then $\phi(X)K[X] \cap R[X] = I^R_{\phi(X)}\phi(X)R[X]$ in [KY7]. In the above proof of Lemma 11.1.5, we used a little more generalized form of this result. ∎

In the next proposition, we shall give a sufficient condition for $\eta(X)$ to satisfy the condition (F).

Proposition 11.1.7 *If* $\mathrm{grade}((I^R_{\phi(X)} \cap I^R_{\psi(X)})\mathsf{c}(\phi(X))) > 1$, *then $\eta(X)$ satisfies the condition* (F) *over R. Consequently, $R[X] :_{R[X]} \eta(X) = (I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X]$.*

Proof By Proposition 11.1.2, we have only to show that $R[X] :_{R[X]} \eta(X) = (I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X]$. The inclusion $\supseteq$ is obvious. We shall show the converse inclusion. Take $f(X) \in R[X] :_{R[X]} \eta(X)$. Then $g(X) := f(X)\eta(X) \in R[X]$. Hence $f(X)\psi(X) = g(X)\phi(X)$. Since $\phi(X)$ and $\psi(X)$ are co-prime in $K[X]$, we have $f(X) \in \phi(X)K[X] \cap R[X]$. Since $\mathrm{grade}((I^R_{\phi(X)} \cap I^R_{\psi(X)})\mathsf{c}(\phi(X))) > 1$, we have $\phi(X)K[X] \cap R[X] = (I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X]$ by Lemma 11.1.5. ∎

Proposition 11.1.8 *If* $\mathrm{grade}((I^R_{\phi(X)} \cap I^R_{\psi(X)})\mathsf{c}(\psi(X))) > 1$, *then $\eta(X)$ satisfies the condition* (F) *over R.*

Proof By Proposition 11.1.2, we need to show that $R[X] :_{R[X]} \eta(X) = (I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X]$. Take $f(X) \in R[X] :_{R[X]} \eta(X)$. Then $f(X)\eta(X) =: g(X) \in R[X]$. Since $f(X)\psi(X) = g(X)\phi(X)$ and $(\phi(X), \psi(X))K[X] = K[X]$, we have $g(X) \in \psi(X)K[X] \cap R[X]$. It follows that $\psi(X)K[X] \cap$

$R[X] = (I^R_{\phi(X)} \cap I^R_{\psi(X)})\psi(X)R[X]$ by Lemma 11.1.5 because $\mathrm{grade}((I^R_{\phi(X)} \cap I^R_{\psi(X)})\mathsf{c}(\psi(X))) > 1$. Thus $g(X) = b\psi(X)j(X)$ for some $b \in I^R_{\phi(X)} \cap I^R_{\psi(X)}$, $j(X) \in R[X]$. So substituting this in $f(X)\psi(X) = g(X)\phi(X)$, we obtain that $f(X) = b\phi(X)j(X) \in (I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X]$. ∎

Let A be a Noetherian domain and $A[Y]$ a polynomial ring. Let α be a nonzero element of an algebraic field extension L of the quotient field $K(A)$ of A and $\pi^{(A)} : A[Y] \to A[\alpha]$ be the A-algebra homomorphism sending Y to α. Let $\varphi_\alpha(Y)^{(A)}$ be the monic minimal polynomial of α over $K(A)$ with $\deg \varphi_\alpha(Y)^{(A)} = d$ and write

$$\varphi_\alpha(Y)^{(A)} = Y^d + \eta_1 Y^{d-1} + \cdots + \eta_d$$

Then η_i $(1 \le i \le d)$ are uniquely determined by α. Let $I^A_{\eta_i} := A :_A \eta_i$ and $I^A_{[\alpha]} := \bigcap_{i=1}^d I^A_{\eta_i}$, the latter of which is called a *generalized denominator ideal* of α. We say that α is an *anti-integral* element if and only if $\mathrm{Ker}(\pi^{(A)}) = I^A_{[\alpha]}\varphi^{(A)}_\alpha(Y)A[Y]$. The concept of anti-integrality is given in Chapter 1 in the birational case, and the higher degree case appears in Chapter 2. For $f(Y) \in A[Y]$, let $\mathsf{c}(f(Y))$ denote the ideal of A generated by the coefficients of $f(Y)$. For an ideal J of $A[Y]$, let $\mathsf{c}(J)$ denote the ideal generated by the coefficients of the elements in J. If α is an anti-integral element over A, then $\mathsf{c}(\mathrm{Ker}(\pi^{(A)})) = \mathsf{c}(I^A_{[\alpha]}\varphi^{(A)}_\alpha(Y)A[Y]) = I^A_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Put $J^A_{[\alpha]} = I^A_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. If $J^A_{[\alpha]} \not\subseteq p$ for all $p \in \mathrm{Dp}_1(A) := \{p \in \mathrm{Spec}(A) \mid \mathrm{depth}(R_p) = 1\}$, then α is called a *super-primitive* element over A. It is known that a super-primitive element is an anti-integral element (*cf.* Chapter 2).

Remark 11.1.9 In the above notation, consider the case: $A = R[X]$ and $\alpha = \eta(X)$. Then $J^A_{[\eta(X)]} = I^A_{\eta(X)} + \eta(X)I^A_{\eta(X)} = I^A_{\eta(X)}(1, \eta(X))R[X]$, where $I^A_{[\eta(X)]} = I^A_{\eta(X)} := R[X] :_{R[X]} \eta(X)$. If $\eta(X)$ satisfies the condition (F), we have $J^A_{[\eta(X)]} = (I^R_{\phi(X)} \cap I^R_{\psi(X)})(\phi(X), \psi(X))R[X]$ because $I^A_{\eta(X)} = (I^R_{\phi(X)} \cap I^R_{\psi(X)})\phi(X)R[X]$ by Proposition 11.1.2. ∎

Theorem 11.1.10 *Assume that*

$$\mathrm{grade}((I^R_{\phi(X)} \cap I^R_{\psi(X)})\mathsf{c}(\phi(X)) + (I^R_{\phi(X)} \cap I^R_{\psi(X)})\mathsf{c}(\psi(X))) > 1$$

Then

(1) $\eta(X)$ *satisfies the condition* (F) *over* R*;*

(2) $\eta(X)$ *is a super-primitive element over* $R[X]$*;*

(3) $((I^R_{\phi(X)} \cap I^R_{\psi(X)})(\phi(X), \psi(X)))R[X] = R[X]$ *if and only if* $R[X, \eta(X)]$ *is flat over* $R[X]$.

Proof (1) For each $p \in \mathrm{Dp}_1(R)$, either $(I^R_{\phi(X)} \cap I^R_{\psi(X)})\mathsf{c}(\phi(X)) \not\subseteq p$ or $(I^R_{\phi(X)} \cap I^R_{\psi(X)})\mathsf{c}(\psi(X)) \not\subseteq p$. In the former case (respectively the latter case), $\eta(X)$ satisfies the condition (F) over R_p by proposition 11.1.7 (respectively Proposition 11.1.8). Thus we have our conclusion by Proposition 11.1.4.
(2) By (1), $\eta(X)$ satisfies the condition (F) over R. So we have $J^{R[X]}_{[\eta(X)]} = (I^R_{\phi(X)} \cap I^R_{\psi(X)})(\phi(X), \psi(X))R[X]$ by the preceding remark. Hence $J^{R[X]}_{[\eta(X)]} \not\subseteq pR[X]$ for each $p \in \mathrm{Dp}_1(R)$. Considering $(\phi(X), \psi(X))K[X] = K[X]$, we have $J^{R[X]}_{[\eta(X)]} \not\subseteq P$ for each $P \in \mathrm{Dp}_1(R[X])$ with $P \cap R = (0)$. Therefore we conclude that $\mathrm{grade}(J_{[\eta(X)]}) > 1$.
(3) By Theorem 2.3.6, we see that $R[X, \eta(X)]$ is flat over $R[X]$ if and only if $J_{[\eta(X)]} = R[X]$. Since $J^{R[X]}_{[\eta(X)]} = (I^R_{\phi(X)} \cap I^R_{\psi(X)})(\phi(X), \psi(X))R[X]$ by the preceding remark, we have our conclusion. ∎

For the next theorem, we introduce the notation: Let α be an algebraic element over $K(X)$ and the minimal (monic) polynomial of α be $\varphi_\alpha(Y) = Y^d + \eta_1(X)Y^{d-1} + \cdots + \eta_d(X)$ with $\eta_i(X) \in K(X)$ and $\eta_0(X) = 1$. Put $\eta_i(X) = \psi_i(X)/\phi_i(X)$, where $\phi_i(X), \psi_i(X) \in K[X]$ with $(\phi_i(X), \psi_i(X))K[X] = K[X]$ and $\phi_i(X)$ is monic.

Theorem 11.1.11 *Under the above circumstances, assume that each $\eta_i(X)$ $(0 \le i \le d)$ satisfies the condition* (F).

(1) *If $\mathrm{grade}(\sum_{j=0}^d(\bigcap_{i=1}^d(I^R_{\phi_i(X)} \cap I^R_{\psi_i(X)})\phi_i(X))\eta_j(X)R[X]) > 1$, then α is a super-primitive element over $R[X]$.*
(2) *If $\sum_{j=0}^d(\bigcap_{i=1}^d(I^R_{\phi_i(X)} \cap I^R_{\psi_i(X)})\phi_i(X))\eta_j(X)R[X] = R[X]$, then $R[X, \alpha]$ is flat over $R[X]$.*

Proof Since each $\eta_i(X)(0 \le i \le d)$ satisfies the condition (F), we have $I^{R[X]}_{\eta_i(X)} = (I^R_{\phi_i(X)} \cap I^R_{\psi_i(X)})\phi_i(X)R[X]$ by Proposition 11.1.2. Thus $I^{R[X]}_{[\alpha]} = \bigcap_{i=1}^d I^{R[X]}_{\eta_i(X)} = \bigcap_{i=1}^d(I^R_{\phi_i(X)} \cap I^R_{\psi_i(X)})\phi_i(X)R[X]$. Since $I^{R[X]}_{[\alpha]}(1, \eta_1(X), \ldots, \eta_d(X)) = J_{[\alpha]}$, we have $\mathrm{grade}(J^{R[X]}_{[\alpha]}) > 1$, which means that α is super-primitive over $R[X]$. Since the flat obstruction ideal of an anti-integral extension $R[X][\alpha]$ of $R[X]$ is $J^{R[X]}_{[\alpha]}$, the last part follows. (See the proof of Theorem 11.1.10(3)). ∎

11.2 Subrings of $R[\alpha]$ Associated with Ideals of R

Assume now that α is an anti-integral element of degree $d \ge 2$ over R. We have seen in Theorem 3.1.18 that $R\langle\alpha\rangle = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ as an R-module, where $\zeta_i := \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i$ $(1 \le i \le d)$. Note that

$\zeta_d = \varphi_\alpha(\alpha) = 0$. Put $\zeta_0 := 1$ and $\eta_0 := 1$ for convenience. It is easy to see that $\alpha\zeta_i = \zeta_{i+1} - \eta_{i+1}$ $(0 \le i \le d-1)$ by definition. Note that $R\langle\alpha\rangle$ and $R[\alpha]$ are birational and their quotient fields are equal to $K(\alpha)$. Let H be an ideal of R and put $C_H := R + H\zeta_1 + \cdots + H\zeta_{d-1}$, which is an R-submodule in $K(\alpha) = K \oplus K\zeta_1 \oplus \cdots \oplus K\zeta_{d-1}$.

The following lemma is seen immediately.

Lemma 11.2.1 *Assume that α is an anti-integral element of degree d over R. Let H denote an ideal of R. Then C_H forms a subring of $K(\alpha)$ if and only if $H^2\zeta_i\zeta_j \subseteq C_H$ for all $i, j \in \{1, 2, \ldots, d-1\}$ with $j \le i$.*

Proposition 11.2.2 *Let H denote an ideal of R. Assume that α is an anti-integral element over R of degree $d = 2$. Then C_H forms a subring of $K(\alpha)$ if and only if $H^2\eta_1 \subseteq H$ and $H^2\eta_2 \subseteq R$.*

Proof Note $\zeta_1 = \alpha + \eta_1$ and hence $\zeta_1^2 = \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1 = \eta_1\zeta_1 - \eta_2$ (here $\zeta_2 = 0$). Assume that C_H is a subring of $K(\alpha)$. Then $H^2\zeta_1^2 \in C_H$ yields that for any $h \in H$, $h^2\eta_1\zeta_1 - h^2\eta_2 \in R + H\zeta_1 \subseteq K \oplus K\zeta_1$. Consequently, $H^2\eta_1 \subseteq H$ and $H^2\eta_2 \subseteq R$. Conversely, take $a + h_1\zeta_1, b + h_2\zeta_1 \in C_H$ with $h_1, h_2 \in H$ and $a, b \in R$. Then $(a + h_1\zeta_1)(b + h_2\zeta_1) = ab + (ah_2 + bh_1)\zeta_1 + h_1h_2\zeta_1^2 \in R + H\zeta_1$ because $h_1h_2\zeta_1^2 \in H^2\zeta_1^2 = H^2(\eta_1\zeta_1 - \eta_2) \subseteq R + H\zeta_1 = C_H$. $\square$

We consider the case $d = 3$, and compute $\zeta_i\zeta_j$ by a definite calculation.

Example 11.2.3 Assume that α is an anti-integral element of degree $d = 3$ over R. Note $\zeta_1 = \alpha + \eta_1$, $\zeta_2 = \alpha\zeta_1 + \eta_2$ and $\zeta_3 = \alpha\zeta_2 + \eta_3$. Thus

$$\begin{aligned}
&\zeta_1^2 = \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1,\\
&\zeta_2\zeta_1 = \zeta_2(\alpha + \eta_1) = \alpha\zeta_2 + \zeta_2\eta_1 = \zeta_3 - \eta_3 + \eta_1\zeta_2 = -\eta_3 + \eta_1\zeta_2, \text{ and}\\
&\zeta_2^2 = \zeta_2(\alpha\zeta_1 + \eta_2) = (\alpha\zeta_2)\zeta_1 + \zeta_2\eta_2 = (\zeta_3 - \eta_3)\zeta_1\\
&\qquad + \zeta_2\eta_2 = -\eta_3\zeta_1 + \eta_2\zeta_2
\end{aligned}$$

because $\zeta_3 = 0$.

Assume that C_H is a subring of $K(\alpha)$. Then we have $H^2\eta_1 \subseteq H$, $H^2\eta_2 \subseteq H$, and $H^2\eta_3 \subseteq H$ since ζ_1^2, $\zeta_2\zeta_1$, and ζ_2^2 are in C_H.

Conversely, if $H^2\eta_1 \subseteq H$, $H^2\eta_2 \subseteq H$ and $H^2\eta_3 \subseteq H$, then ζ_1^2, $\zeta_2\zeta_1$ and ζ_2^2 are in C_H. So we conclude that C_H is a subring of $K(\alpha)$ by Lemma 11.2.1.

Lemma 11.2.4 *Assume that α is an anti-integral element of degree $d(\geq 3)$ over R and that $1 \leq j \leq i < d$.*

(i) *If $i + j < d$, then*

$$\zeta_i\zeta_j = \zeta_{i+j} - \sum_{t=1}^{j} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s}$$

(ii) *If $i + j \geq d$, then*

$$\zeta_i\zeta_j = -\sum_{t=1}^{d-i} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{d-i-1} \eta_{j-s}\zeta_{i+s}$$

(Note that $d - i > 0$ and $d - i - 1 \geq 0$ because $1 \leq j \leq i < d$.)

Proof Note first that $\eta_0 := 1$, $\zeta_0 := 1$, $\zeta_d = 0$, and $\zeta_{i+1} = \alpha\zeta_i + \eta_{i+1}$. Now we compute $\zeta_i\zeta_j$ as follows:

$$\begin{aligned}
\zeta_i\zeta_j &= \zeta_i(\alpha\zeta_{j-1} + \eta_j) \\
&= \alpha\zeta_i\zeta_{j-1} + \eta_j\zeta_i \\
&= (\zeta_{i+1} - \eta_{i+1})\zeta_{j-1} + \eta_j\zeta_i \\
&= \zeta_{i+1}\zeta_{j-1} - \eta_{i+1}\zeta_{j-1} + \eta_j\zeta_i \\
&= \zeta_{i+2}\zeta_{i-2} - \eta_{i+2}\zeta_{j-2} + \eta_{j-1}\zeta_{i+1} - \eta_{i+1}\zeta_{j-1} + \eta_j\zeta_i \\
&= \zeta_{i+2}\zeta_{i-2} - (\eta_{i+2}\zeta_{j-2} + \eta_{i+1}\zeta_{j-1}) + (\eta_j\zeta_i + \eta_{j-1}\zeta_{i+1}) \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots
\end{aligned}$$

(i) Repeating the above process, we have

$$\zeta_i\zeta_j = \zeta_{i+j}\zeta_0 - \sum_{t=1}^{j} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s}$$

that is,

$$\zeta_i\zeta_j = \zeta_{i+j} - \sum_{t=1}^{j} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s}$$

(ii) Put $\ell := i + j - d$. Then $j \leq i < d$ yields $\ell < d - 1$ and $j - \ell \geq 1$. Thus continuing the above process yields

$$\zeta_i\zeta_j = \zeta_d\zeta_\ell - \sum_{t=1}^{j-\ell} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-\ell-1} \eta_{j-s}\zeta_{i+s}$$

that is,

$$\zeta_i\zeta_j = -\sum_{t=1}^{d-i}\eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{d-i-1}\eta_{j-s}\zeta_{i+s} \qquad \square$$

Remark 11.2.5 Assume that α is an anti-integral element of degree $d(\geq 3)$ over R and that $1 \leq j \leq i < d$. For (i, j) with $i+j < d$, let $\Delta_1^{(j)} := \{j-t \mid t = 1, \ldots, j\}$ and $\Delta_1^{(i,j)} := \{i+s \mid s = 0, \ldots, j-1\}$. For (i, j) with $i+j \geq d$, let $\Delta_2^{(j,i)} := \{j-t \mid t = 1, \ldots, d-i\}$ and $\Delta_2^{(i)} := \{i+s \mid s = 0, \ldots, d-i-1\}$.

(i) If $i+j < d$, then $\Delta_1^{(i,j)} \cap \Delta_1^{(j)} = \emptyset$ because $j-t < i+s$, and $\bigcup_{i+j<d}(\Delta_1^{(i,j)} \cup \Delta_1^{(j)}) = \{0, 1, 2, \ldots, d-1\}$.

(ii) Assume $i+j \geq d$. Then $\Delta_2^{(i)} \cap \Delta_2^{(j,i)} = \emptyset$ because $j-t < i+s$. Put $e_k := -1$ if $k \in \Delta_2^{(j,i)}$ and $e_k := 1$ if $k \in \Delta_2^{(i)}$. Then

$$\zeta_i\zeta_j = \sum_{k\in\Delta_2^{(j,i)}\cup\Delta_2^{(i)}} e_k\eta_{i+j-k}\zeta_k$$

Consider ζ_{d-1}^2. Then we have

$$\begin{aligned}\zeta_{d-1}^2 &= \zeta_{d-1}(\alpha\zeta_{d-2} + \eta_{d-1})\\ &= \alpha\zeta_{d-1}\zeta_{d-2} + \eta_{d-1}\zeta_{d-1}\\ &= (\zeta_d - \eta_d)\zeta_{d-2} + \eta_{d-1}\zeta_{d-1}\\ &= -\eta_d\zeta_{d-2} + \eta_{d-1}\zeta_{d-1}\end{aligned}$$

Hence if $\zeta_{d-1}^2 \in C_H$, then $H^2\eta_d \subseteq H$. $\square$

Theorem 11.2.6 *Assume that α is an anti-integral element of degree $d(\geq 3)$ over R. Let H be an ideal of R. Then $C_H := R + H\zeta_1 + \cdots + H\zeta_{d-1}$ is a subring of $K(\alpha)$ if and only if $H^2\eta_i \subseteq H$ for all i $(1 \leq i \leq d)$.*

Proof Our conclusion follows Lemmas 11.2.1 and 11.2.4 and Remark 11.2.5. $\square$

Corollary 11.2.7 *Assume that α is an anti-integral element of degree $d(\geq 3)$ over R. Let H be an ideal of R. If C_H is a subring of $K(\alpha)$, then every element in $H\eta_i$ is integral over R for all i $(1 \leq i \leq d)$.*

Proof We have $H^2\eta_i \subseteq H$ for all i $(1 \leq i \leq d)$ by Theorem 11.2.6. So $H\eta_i \subseteq H :_K H$. Since H is a finitely generated ideal, any element in $H :_K$

H is integral over R. Thus every element of $H\eta_i$ is integral over R for all i $(1 \leq i \leq d)$. $\square$

Corollary 11.2.8 *Assume that R is normal and that α is of degree d (≥ 3) over R. Let H be an ideal of R. If C_H is a subring of $K(\alpha)$, then $H \subseteq I_{[\alpha]}$ and hence $C_H \subseteq R\langle\alpha\rangle$.*

Proof Since R is normal, the element α is anti-integral over R by Theorem 2.2.9. We have $H\eta_i \subseteq H :_K H = R$ for all i $(1 \leq i \leq d)$, that is, $H \subseteq I_{[\alpha]}$. $\square$

Theorem 11.2.9 *Let H be an ideal of R. Assume that α is an anti-integral element of degree d (≥ 2) over R.*

(1) *If C_H is a subring of $K(\alpha)$, then $H^2 \subseteq I_{[\alpha]}$.*
(2) *Assume that* $\mathrm{grade}(H) > 1$. *Then C_H is a subring of $K(\alpha)$ if and only if $\eta_i \in R$ for all i, if and only if α is integral over R.*
(3) *If $I_{[\alpha]} \neq R$ and if C_H is a subring of $K(\alpha)$, then* $\mathrm{grade}(H) \leq 1$.
(4) *When H is an invertible ideal and $d \geq 3$, $H \subseteq I_{[\alpha]}$ if and only if C_H is a subring of $K(\alpha)$.*

Proof (1) Assume that $d \geq 3$. Since C_H is a ring, we have $H^2\eta_i \subseteq H$ for all $(1 \leq i \leq d)$ by Theorem 11.2.6. Hence $H^2\eta_i \subseteq H \subseteq R$, which means that $H^2 \subseteq I_{\eta_i}$ for all $(1 \leq i \leq d)$. Thus $H^2 \subseteq I_{[\alpha]}$. Next assume that $d = 2$. Then Proposition 11.2.2 yields our conclusion.
(2) Assume that C_H is a subring of $K(\alpha)$. Then $H^2 \subseteq I_{[\alpha]}$ by (1). Hence $1 \leq \mathrm{grade}(H) \leq \mathrm{grade}(I_{[\alpha]})$, which means that $I_{[\alpha]} = R$ because $I_{[\alpha]}$ is a divisorial ideal of R. So we have $I_{\eta_i} = R$ for all i $(1 \leq i \leq d)$, that is, $\eta_i \in R$ for all i $(1 \leq i \leq d)$. Furthermore α is integral over R by Theorem 2.3.2. Conversely, assume that α is integral over R. Then $I_{[\alpha]} = R$, and hence $\eta_i \in R$ for all $(1 \leq i \leq d)$. So, if $d \geq 3$, $H^2\eta_i \subseteq H$ yields that C_H is a ring by Theorem 11.2.6. If $d = 2$, C_H is a ring by Proposition 11.2.2.
(3) Our conclusion follows from the result (2).
(4) Since $d \geq 3$, C_H is a subring if and only if $H^2\eta_i \subseteq H$ for all i $(1 \leq i \leq d)$. Since H is invertible, $H^2\eta_i \subseteq H$ for all i $(1 \leq i \leq d)$ if and only if $H\eta_i \subseteq R$ for all i $(1 \leq i \leq d)$ if and only if $H \subseteq I_{[\alpha]}$. $\square$

Theorem 11.2.10 *Assume that α is an anti-integral element of degree $d(\geq 2)$ over R. Let H be an ideal of R. If $H \subseteq I_{[\alpha]}$, then C_H is a subring of $K(\alpha)$.*

Proof The inclusion $H \subseteq I_{[\alpha]}$ yields that $H^2\eta_i \subseteq H(\subseteq R)$ for all i $(1 \leq i \leq d)$. Hence C_H is a ring by Proposition 11.2.2 and Theorem 11.2.6. ∎

Corollary 11.2.11 *Assume that R is a locally factorial domain and that α is an element of degree $d(\geq 2)$. The collection $\Delta := \{C_H \mid H$ is an ideal of R and C_H is a subring of $K(\alpha)\}$ has the maximum member $C_{I_{[\alpha]}}(= R\langle\alpha\rangle)$.*

Proof Note first that α is anti-integral over R because R is a Krull domain (*cf.* Theorem 2.2.9). Take $C_H \in \Delta$. Then $H^2 \subseteq I_{[\alpha]}$ by Theorem 11.2.9(1). Since $I_{[\alpha]}$ is a divisorial ideal, we have $\mathrm{grade}(H) = \mathrm{grade}(H^2) = 1$. Since R is locally factorial, every nonzero ideal of grade 1 is invertible. So H is invertible. Hence $H \subseteq I_{[\alpha]}$ by Theorem 11.2.9(4). ∎

Finally we close this section by giving the following results concerned with the ring $R\langle\alpha\rangle$.

Proposition 11.2.12 *Assume that α is an anti-integral element of degree d. If an element a in R is a non-zero-divisor on $R/I_{[\alpha]}$, then $R[a\alpha] \cap R[\alpha^{-1}] = R + I_{[\alpha]}(a\zeta_1) + \cdots + I_{[\alpha]}(a^{d-1}\zeta_{d-1})$.*

Proof We have only to show the inclusion $(\subseteq)$ because $a^i I_{[\alpha]}\zeta_i \subseteq R[a\alpha] \cap R[\alpha^{-1}]$. Take an element $\beta \in R[a\alpha] \cap R[\alpha^{-1}]$ and write $\beta = x_n(a\alpha)^n + \cdots + x_1(a\alpha) + x_0 = y_0 + y_1(\alpha^{-1}) + \cdots + y_m(\alpha^{-1})^m$ with $x_i, y_j \in R$. Then we have $\alpha^m\beta = x_n a^n \alpha^{n+m} + \cdots + x_0\alpha^m = y_0\alpha^m + y_1\alpha^{m-1} + \cdots + y_m$. Put $f(X) := x_n a^n X^{n+m} + \cdots + x_0 X^m - (y_0 X^m + y_1 X^{m-1} + \cdots + y_m) \in \mathrm{Ker}(\pi)$, where $\pi : R[X] \to R[\alpha]$ denotes the canonical R-homomorphism. Since α is an anti-integral element over R, $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Hence $x_n a^n \in I_{[\alpha]}$. Since a is a nonzero-divisor of $R/I_{[\alpha]}$, we have $x_n \in I_{[\alpha]}$. Put $\beta_*(X) := x_n a^n X^n + \cdots + x_1 a X + x_0$. Since $x_n\varphi_\alpha(X) \in R[X]$, considering $\deg(\beta_*(X) - x_n\varphi_\alpha(X)X^{n-d}) < n$ if $n \geq d$, we can use the induction on n and assume that $n < d$. So, considering $\beta - x_n a^n \zeta_n$ instead of β, we can conclude that $\beta \in R + I_{[\alpha]}(a\zeta_1) + \cdots + I_{[\alpha]}(a^{d-1}\zeta_{d-1})$ by induction on n. ∎

Proposition 11.2.13 *Assume that α is an anti-integral element of degree d over R. If an element a in R is a non-zero-divisor on $R/I_{[\alpha^{-1}]}$, then $R[a^{-1}\alpha] \cap R[\alpha^{-1}] = R\langle\alpha\rangle$.*

Proof We have only to show the inclusion $(\subseteq)$. Take an element $\beta \in R[a^{-1}\alpha] \cap R[\alpha^{-1}]$ and write $\beta = x_n(a^{-1}\alpha)^n + \cdots + x_1(a^{-1}\alpha) + x_0 = y_0 + y_1(\alpha^{-1}) + \cdots$

$+ y_m(\alpha^{-1})^m$ with $x_i, y_j \in R$. Then $a^n\beta = x_n\alpha^n + \cdots + a^n x_0 = a^n y_0 + a^n y_1(\alpha^{-1}) + \cdots + a^n y_m(\alpha^{-1})^m$. Since α is anti-integral over R, so is α^{-1} (*cf.* Theorem 2.4.5). Putting $f(X) := a^n y_m X^{m+n} + \cdots + a^n X^n - (a^n x_0 X^n + \cdots + x_n)$, we have $f(\alpha^{-1}) = 0$ and hence $f(X) \in I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(X)$. Thus we have $a^n y_m \in I_{[\alpha^{-1}]}$. Since a is a nonzero-divisor on $R/I_{[\alpha^{-1}]}$, we have $y_m \in I_{[\alpha^{-1}]}$. Hence $y_m\varphi_{\alpha^{-1}}(X) \in R[X]$. Put $\beta_*(X) := y_0 + y_1 X + \cdots + y_m X^m$. Considering $\deg(\beta_*(X) - y_m\varphi_{\alpha^{-1}}(X)) < m$, we can assume that $m < d$ by induction. Let $\varphi_{\alpha^{-1}}(X) := X^d + \eta_1' X^{d-1} + \cdots + \eta_d'$ be the monic minimal polynomial of α^{-1} over K and $\zeta_i' := (\alpha^{-1})^i + \eta_1'(\alpha^{-1})^{i-1} + \cdots + \eta_i'$ $(1 \le i \le d)$. Considering $\beta - y_m\zeta_m'$ instead of β, we obtain that $\beta - y_m\zeta_m' \in R + I_{[\alpha^{-1}]}\zeta_1' + \cdots + I_{[\alpha^{-1}]}\zeta_{d-1}' = R[\alpha^{-1}] \cap R[\alpha]$ (*cf.* Theorem 3.1.18) by induction on m. Thus $\beta \in R[\alpha^{-1}] \cap R[\alpha]$. Therefore we have $R[a^{-1}\alpha] \cap R[\alpha^{-1}] \subseteq R\langle\alpha\rangle$. $\square$

11.3 Semi Anti-Integral Elements

Let R be a Noetherian domain with quotient field k. Let L be an extension of a field K and α be an element of L which is algebraic over K. Put $d = [K(\alpha) : K]$. We consider the canonical exact sequence

$$0 \longrightarrow I \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0.$$

If I is generated by polynomials of degree d, then α is said to be an anti-integral element over R (*cf.* Proposition 2.2.3). Let $A = R[\alpha]$. It is well known that if α is integral and anti-integral over R, then A is a free R-module of rank d.

Note that α has a unique monic relation of degree d over K. Let $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ $(\eta_i \in K)$ be the minimal polynomial of α over K and $I_{\eta_i} = R :_R \eta_i = \{a \in R \mid a\eta_i \in R\}$ and $I_{[\alpha]} = \bigcap_{i=1}^d I_{\eta_i}$. Then we have:

Proposition 11.3.1 *The following statements are equivalent:*

(1) *α is an anti-integral element over R;*

(2) $R[X] :_{K[X]} \varphi_\alpha(X) \subseteq I_{[\alpha]}R[X]$.

Proof (1) $\Rightarrow$ (2): For any element $g(X)$ of $R[X] :_{K[X]} \varphi_\alpha(X$, we have $g(X)\varphi_\alpha(X) = f(X)$ for some $f(X) \in R[X]$. Let $\pi : R[X] \longrightarrow R[\alpha]$ be the natural ring homomorphism. Then $f(X) \in \mathrm{Ker}(\pi)$. We consider the natural exact sequence

$$0 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0.$$

Since α is anti-integral over R, $\mathrm{Ker}(\pi) = I_{[\alpha]}$, and hence $f(X) = \sum_i (h_i \varphi_\alpha(X))$ $t_i(X)$ for some $t_i(X) \in I_{[\alpha]}$, $t_i(X) \in R[X]$. Canceling $\varphi_\alpha(X)$, we see that $g(X) = \sum_i h_i t_i(X) \in I_{[\alpha]} R[X]$.
(2) $\Rightarrow$ (1): For any element $f(X) \in \mathrm{Ker}(\pi)$, we have $f(X) = g(X)\varphi_\alpha(X)$ for some $g(X) \in K[X]$. Thus $g(X) \in R[X] :_{K[X]} \varphi_\alpha(X) \subseteq I_{[\alpha]} R[X]$ by assumption. Hence we can write $g(X) = \sum_i h_i t_i(X)$, $h_i \in I_{[\alpha]}$, $t_i(X) \in R[X]$. Therefore we obtain $f(X) = \sum_i (h_i \varphi_\alpha(X)) t_i(X) \in I_{[\alpha]} \varphi_\alpha(X) R[X]$. Thus α is anti-integral over R. $\square$

This observation leads to the following definition.

Definition 11.3.2 If $R[X] :_{K[X]} \varphi_\alpha(X) \subseteq R[X]$, α is said to be a *semi anti-integral* element over R.

If α is an anti-integral element over R, it is a semi anti-integral element over R by Proposition 11.3.1.

Remark 11.3.3 Let $\overline{R}$ be the integral closure of R in K. By Gauss' Lemma, we have easily seen $\eta_i \in \overline{R}$ for all i, if α is integral over R. $\square$

Next we shall show that if α is an integral and a semi anti-integral element over R, then α is an anti-integral element over R. For the purpose of this proof, we need the following proposition. If α is an anti-integral element over R, we put

$$\mathbf{H}_\alpha := R[X] :_{K[X]} \varphi_\alpha(X)$$

Proposition 11.3.4 *The following statements are equivalent:*

(1) *α is an anti-integral element over R;*

(2) $\boldsymbol{H}_\alpha = I_{[\alpha]} R[X]$.

Proof (1) $\Rightarrow$ (2): Let $g(X)$ be an arbitrary element of $\mathbf{H}_\alpha$. Then, from the proof (1) $\Rightarrow$ (2) of Proposition 11.3.1, we can write $g(X) = \sum_i H_i t_i(X)$, $h_i \in I_{[\alpha]}$, $t_i(X) \in R[X]$. Hence we see that $g(X) \in I_{[\alpha]} R[X]$. Since one inclusion is obvious, we have the desired result.
(2) $\Rightarrow$ (1): Assume that $f(\alpha) = 0$ for some $f(X) \in R[X]$. Then we have $f(X) = g(X)\varphi_\alpha(X)$ for some $g(X) \in K[X]$. Thus $g(X) \in R[X] :_{K[X]} \varphi_\alpha(X) = \mathbf{H}_\alpha$. Hence we have $g(X) = \sum_i g_i X^i$, with $g_i \in I_{[\alpha]}$. Note that $g_i \varphi_\alpha(X) \in R[X]$ and $\deg(g_i \varphi_\alpha(X)) = d$ for all i. Therefore $f(X)$ is generated by $I_{[\alpha]} \varphi_\alpha(X)$. Thus α is anti-integral over R. $\square$

As a consequence we have the following result.

Theorem 11.3.5 *If α is an integral and a semi anti-integral element over R, then α is an anti-integral element over R, and hence $A = R[\alpha]$ is a free R-module of rank d.*

Proof Since α is integral over R, there exists a monic polynomial $f(X) \in R[X]$ such that $f(\alpha) = 0$. First, we consider the case of $\deg f(X) = d$. In this case, α is anti-integral over R, and hence there is nothing to prove. Next we assume that $n = \deg f(X) > d$. Let

$$f(X) = (X^{n-d} + \lambda_1 X^{n-d-1} + \cdots + \lambda_{n-d})\varphi_\alpha(X)$$

$$= (X^{n-d} + \lambda_1 X^{n-d-1} + \cdots + \lambda_{n-d})(X^d + \eta_1 X^{d-1} + \cdots + \eta_d)$$

$\lambda_i, \eta_j \in K$. Since α is semi anti-integral over R, we see $\lambda_1, \lambda_2, \ldots, \lambda_{n-d} \in R$ by the definition. If we compare the coefficients of X^{n-1} in both sides, we see that $\eta_i + \lambda_j \in R$, and hence $\eta_i \in R$. Proceeding in this way, we get $\eta_i \in R$ for all i. Thus $\varphi_\alpha(X) \in R[X]$, so that α is anti-integral over R. ∎

Remark 11.3.6 Let R be a Noetherian domain with quotient field K. Let L be a field-extension of K and α be an element of L which is algebraic over K. Put $d = [K(\alpha) : K]$. Assume that α is integral over R. Then there exists a monic polynomial $f(X) \in R[X]$ such that $f(\alpha) = 0$. Then we have:
$\deg f(X) = d \Leftrightarrow R[\alpha]$ is a free R-module of rank $d \Leftrightarrow \varphi_\alpha(X) \in R[X] \Leftrightarrow f(X) = \varphi_\alpha(X)$.
Thus we obtain the following proposition. ∎

Proposition 11.3.7 $\{p \in \mathrm{Spec}(R) \mid R_p[X]$ *is not flat over* $R_p\} = \{p \in \mathrm{Spec}(R) \mid p \supseteq I_{[\alpha]}\}$.

We assume that there exists a monic polynomial $f(X) \in R[X]$ of degree $d+1$ such that $f(\alpha) = 0$. Then we have $f(X) = (X + \lambda)\varphi_\alpha(X)$, $\lambda \in K$. It follows that $\lambda \in \overline{R}$. Thus we have the following proposition.

Proposition 11.3.8 *The following statements are equivalent:*

(1) $\lambda \in R$;

(2) $\varphi_\alpha(X) \in R[X]$.

Therefore $\{p \in \mathrm{Spec}(R) \mid R_p[\alpha]$ *is not flat over* $R_p\} = \{p \in \mathrm{Spec}(R) \mid p \supseteq I_\lambda\}$.

Proof Let $f(X) = X^{d+1} + a_1 X^d + \cdots + a_d$ and $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Since $f(X) = (X + \lambda)\varphi_\alpha(X)$, we deduce that $a_1 = \lambda + \eta_1, a_i =$

$\lambda\eta_{i-1} + \eta_i$ $(2 \leq i \leq d)$, and $a_{d+1} = \lambda\eta_d$. Note that $a_i \in R$ $(1 \leq i \leq d)$. Thus we see that $\lambda \in R$ is equivalent to $\eta_i \in R$. The second half is easily seen from the fact that $R_p[\alpha]$ is flat over R_p if and only if $\varphi_\alpha(X) \in R[X]$. This completes the proof. $\square$

Remark 11.3.9 Consider the canonical exact sequence:

$$0 \longrightarrow P \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0$$

Assume that there exists a monic polynomial $f(X) \in R[X]$ of degree $d+1$ with $f(\alpha) = 0$. Then we have $P = I_{[\alpha]}\varphi_\alpha(X)R[X] + f(X)R[X]$. $\square$

Finally, we have the following theorem.

Theorem 11.3.10 *Let R be a Noetherian domain with quotient field K. Let L be an extension of a field K and let α be an element of L which is algebraic over K. Let $d = [K(\alpha) : K]$ and $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$, $\eta_i \in K$ be a monic relation of α over K. Assume that α is integral over R. Let B be an intermediate ring between R and $\overline{R}$. Then the following statements are equivalent:*

(1) $R[X] :_{K[X]} \varphi_\alpha(X) \subseteq B[X]$;

(2) *$B[\alpha]$ is flat over B and α is anti-integral over B.*

11.4 Pseudo-Simple Extensions

Let R be a Noetherian integral domain, K denote the quotient field of R, and A be an R-algebra of finite type contained in K. Then $A = R[F]$ for some fractional ideal F of R. We can write $F = I\alpha$ for some ideal I of R and some element α in K. So A is of the form $R[I\alpha]$. In general, it seems natural to consider an extension like $R[I\alpha]$, where I is an ideal of R and α is an element in some extension of R, where α does not necessarily belong to K. We say that $R[I\alpha]$ is a *pseudo-simple* extension of R defined by I and α if $\operatorname{depth}(R_P) > 1$ for any prime ideal P containing I. Our objective of this section is to study the ring like $R[I\alpha]$. In particular, we are interested in the case α is anti-integral over R. The anti-integrality is studied in Chapters 1, 2, 3, and 4.

Throughout this section, we use the following notations unless otherwise specified: let R be a Noetherian domain, K the quotient field of R, and $\overline{R}$ the

integral closure of R in K. For a nonzero element α of K, put

$$R :_R \alpha := \{a \in R \mid a\alpha \in R\}(= \alpha^{-1}R \cap R)$$

We start with recalling the following definition which is seen in Chapter 1: Let

$$R\langle\alpha\rangle := R[\alpha] \cap R[\alpha^{-1}]$$

in K. We say that α is *anti-integral* over R if $R\langle\alpha\rangle = R$.

Theorem 11.4.1 *Assume that R is a local domain with the maximal ideal m and let α be a nonzero element in K. Assume that α is anti-integral over R and that $R[\alpha] = R[m\alpha]$. Then $\alpha \in R$ or $\alpha^{-1} \in R$.*

Proof Since $\alpha \in R[m\alpha]$, we can write

$$\alpha = b_0 + b_1\alpha + \cdots + b_n\alpha^n$$

where $b_i \in m^i$ for all $i > 0$ and $b_0 \in R$.

Case I: Assume that $b_0 \notin m$. Then $1 - b_0(\alpha^{-1}) = b_1 + b_2\alpha + \cdots + b_n\alpha^{n-1} \in R\langle\alpha\rangle = R\langle\alpha\rangle = R$. Hence $1 - b_0(\alpha^{-1}) \in R$. Since b_0 is a unit in R, we have $\alpha^{-1} \in R$.

Case II: Assume that $b_0 \in m$. Then we have $b_0 + (b_1 - 1)\alpha + \cdots + b_n\alpha^n = 0$. Put $f(X) = b_nX^n + \cdots + b_2X^2 + (b_1 - 1)X + b_0$. Then $b_1 - 1 \notin m$. Thus $\mathsf{c}(f(X)) = R$, where $\mathsf{c}(\ \ )$ denotes the content ideal. Take $h(X) \in R[X]$ which satisfies $h(\alpha) = 0$, $\mathsf{c}(h(X)) = R$, and $\deg h(X)$ is minimal among such ones. Write

$$h(X) = c_dX^d + \cdots + c_0$$

with $c_i \in R$ and

$$c_d\alpha^d + \cdots + c_0 = 0 \qquad (*)$$

If c_0 is a unit in R, then by the same argument as in Case I, we have that $\alpha^{-1} \in R$. Assume that $c_0 \in m$. Then there exists s $(0 < s \leq d)$ such that c_s is a unit in R. From $(*)$, we have $c_0\alpha^{d-s+1} + \cdots + c_s\alpha + c_{s-1} = -(c_{s-2}\alpha^{-1} + \cdots + c_0\alpha^{-s+1}) \in R\langle\alpha\rangle = R$. Thus there exists $v \in R$ such that $c_d\alpha^{d-s+1} + \cdots + c_s\alpha + v = 0$. By the minimality of $\deg h(X)$, we get $d - s + 1 = d$, that is, $s = 1$. Hence c_1 is a unit in R. Consider the equality $c_d\alpha^{d-1} + \cdots + c_1 = -(\alpha^{-1})c_0 \in R\langle\alpha\rangle = R$. Put $t = c_d\alpha^{d-1} + \cdots + c_1 = -(\alpha^{-1})c_0$. If $c_1 - t \notin m$, then the polynomial $c_dX^{d-1} + \cdots + c_1 - t \in R[X]$ gives a contradiction to the minimality of $\deg h(X)$.

So $c_1 - t \in m$. Hence $t \notin m$ because c_1 is a unit in R. Thus $t = -(\alpha^{-1})c_0$ implies $\alpha t + c_0 = 0$ and hence $\alpha \in R$. $\square$

Corollary 11.4.2 *Assume that (R, m) is a normal local domain and let α be a nonzero element in K such that neither α nor α^{-1} belongs to R. Then $R[\alpha] \neq R[m\alpha]$.*

Proof Since R is normal and $R\langle\alpha\rangle$ is integral over R, $R\langle\alpha\rangle = R$, that is, any $\alpha \in K$ is anti-integral over R. So our conclusion follows from Theorem 11.4.1. $\square$

Corollary 11.4.3 *Let R be a Noetherian domain, let I be an ideal of R, and let α be a nonzero element in K which is anti-integral over R. If $R[\alpha] = R[I\alpha]$, then $R_p[\alpha]$ is flat over R_p for any $p \in \mathrm{Spec}(R)$ with $p \supseteq I$.*

Proof We may assume that R is a local domain with the maximal ideal m. Since $R[\alpha] = R[I\alpha]$ implies $R[\alpha] = R[m\alpha]$, $R[\alpha]$ is flat over R by Theorem 11.4.1. $\square$

Corollary 11.4.4 *Assume that (R, m) is a local domain. Then the following statements are equivalent:*

(a) *R is a DVR;*

(b) *every α in $K \setminus \{0\}$ is anti-integral over R and $R[\alpha] = R[m\alpha]$.*

Proof (a) $\Rightarrow$ (b) follows from Corollary 11.4.2.
(b) $\Rightarrow$ (a): For any nonzero element α in K, $\alpha \in R$ or $\alpha^{-1} \in R$ by Theorem 11.4.1, which asserts that R is a valuation domain. Since R is Noetherian, R is a DVR. $\square$

Lemma 11.4.5 *Let I be an ideal of R and P a prime ideal containing I. Then if* $\mathrm{depth}(R_P) > 1$, *then* $\mathrm{grade}(IR_P) > 1$.

Proof We may assume that (R, m) is a local ring. Take $a \in I$, $a \neq 0$ and let $aR = Q_1 \cap \cdots \cap Q_n$ be a primary decomposition with $P_i = \sqrt{Q_i}$. In this case $\mathrm{depth}(R_P) = 1$ for all $P = P_i$. Since $P_i \not\supseteq$I by the assumption, we have $I \not\subseteq P_1 \cup \cdots \cup P_n$. Hence there exists $b \in I \setminus \bigcup P_i$. It is easy to see that $\{a, b\}$ is a regular sequence. Thus $\mathrm{grade}(I) > 1$. $\square$

Definition 11.4.6 Let I be an ideal of R and α be an element which belongs to some extension of R which is not necessarily equal to K. Consider the extension $A = R[I\alpha]$ of R. We say that A is a *pseudo-simple* extension of R defined by I and α if $I = R$ or $\mathrm{grade}(I) > 1$.

Any simple extension $R[\alpha]$ (where α belongs to some extension of R) is a pseudo-simple extension of R.

In the next theorem, we use the following fact: for $\alpha \in K$, $R :_R \alpha$ is a divisorial ideal of R, that is, $R :_K (R :_K (R :_R \alpha)) = R :_R \alpha$. This follows from the facts $\alpha(R :_R \alpha) = \alpha R \cap R$ and $R :_K (R :_R \alpha)$ contains α.

Theorem 11.4.7 *Let α be an element in some extension of R. Assume that $A = R[I\alpha]$ is a pseudo-simple extension of R defined by I and α. If A is flat over R, then $A = R[\alpha]$, that is, A is a simple extension generated by α.*

Proof We may assume that (R, m) is a local ring. By Lemma 11.4.5, there exist $a, b \in I$ which form a regular sequence. It is obvious that $a\alpha,\ b\alpha \in A$. Since A is flat over R, either a, b form an A-regular sequence or $(a, b)A = A$. In the former case, $A :_A \alpha$ contains a, b and hence $A :_A \alpha = A$ because $A :_A \alpha$ is a divisorial ideal of A. Thus $\alpha \in A$. In the latter case, $(a, b)A = A$ implies that $1 = a\beta + b\gamma$ for some $\beta, \gamma \in A$. Thus $\alpha = (a\alpha)\beta + (b\alpha)\gamma \in A$. Therefore in any case, we conclude $\alpha \in A$. $\square$

In the following examples, let k denote a field and $k[x, y]$ denote a polynomial ring.

Example 11.4.8 Let $R = k[x, y]$, $I = (x, y)$ and $\alpha = 1/xy$. Since $\alpha^{-1} \in R$, $R = R\langle\alpha\rangle$ and hence α is anti-integral over R. By definition, $R[I\alpha]$ is a pseudo-simple extension of R and $R[I\alpha] = R[(x, y)(1/xy)] = R[1/x, 1/y] = R[1/xy] = R[\alpha]$, a simple extension. Moreover since $\alpha^{-1} \in R$, $R = R[\alpha^{-1}] \to R[\alpha^{-1}, \alpha] = R[\alpha]$ is obtained by localization and hence is flat. This shows that $R[I\alpha] = R[\alpha]$ does not always imply $I = R$.

Example 11.4.9 Let $R = k[x, y]$ and $A = R[x/y, y/x] = R[(x^2, y^2)(1/xy)]$, which is a pseudo-simple extension of R. It is not hard to see that A is not a simple extension of R. Since $\alpha^{-1} = xy \in R$, α is anti-integral over R. By Theorem 11.4.7, A is not flat over R.

Example 11.4.10 Let I be an ideal (x, y) of $R = k[x, y]$ and α is an element in some extension of R which satisfies $y^2\alpha^2 + (x-1)\alpha + 1 = 0$. Then $[K(\alpha) : K] = 2$, that is, $\alpha \notin K$. $R[I\alpha]$ is a pseudo-simple extension of R and $R[I\alpha] =$

$R[x\alpha, y\alpha]$. Since $\alpha \in R[I\alpha]$, we have $R[I\alpha] = R[\alpha]$. Since α^{-1} is a zero of the single monic polynomial and $\alpha^{-1} \in R[\alpha]$, $R \to R[\alpha^{-1}]$ is flat and $R[\alpha^{-1}] \to R[\alpha^{-1}, \alpha] = R[\alpha]$ is a localization. Hence $R \to R[\alpha]$ is flat.

Lemma 11.4.11 *Let a be an element of R and let α be a nonzero element which is algebraic over R and let $\varphi_\alpha(X) = X^d + \mu_1 X^{d-1} + \cdots + \mu_d$ be the minimal polynomial of α in $K[X]$. Assume that the kernel of the canonical homomorphism $R[X] \to R[\alpha]$ is generated by some polynomials of degree d. Put $I_{[\alpha]} = \bigcap_{1 \le i \le d}(R :_R \mu_i)$. Then*

(i) *if $a \in I_{[\alpha]}$, $a\alpha$ is integral over R;*

(ii) *if $a\alpha$ is integral over R, $a \in \sqrt{I_{[\alpha]}}$.*

Proof (i) Put $b_i = a\mu_i \in R$. Then $a\alpha^d + b_i\alpha^{d-1} + \cdots + b_d = 0$ is a relation of α over R. From this, we get $(a\alpha)^d + ab_i(a\alpha)^{d-1} + \cdots + a^d b_d = 0$. So $a\alpha$ is integral over R.
(ii) Let $(a\alpha)^n + c_1(a\alpha)^{n-1} + \cdots + c_n = 0$ be a relation of $a\alpha$ over R and put $h(X) = a^n X^n + c_1 a^{n-1} X^{n-1} + \cdots + c_n$. Then $h(\alpha) = 0$ and we can write $h(X) = \sum_i f_i(X) g_i(X)$ with $f_i(\alpha) = 0$, $\deg f_i(X) = d$ for all i. Since every coefficient of $f_i(X)$ belongs to $I_{[\alpha]}$, $a^n \in I_{[\alpha]}$. Thus $a \in \sqrt{I_{[\alpha]}}$. ∎

Proposition 11.4.12 *Let α be a nonzero element which is algebraic over R and let $A = R[I\alpha]$ be a pseudo-simple extension of R defined by I and α. Assume that the kernel of the canonical homomorphism $R[X] \to R[\alpha]$ is generated by some polynomials of degree d. If A is integral over R, then α is integral over R.*

Proof We have only to show that α is integral over R_P for all $P \in \mathrm{Spec}(R)$. If P does not contain I, $A_P = R_P[\alpha]$ implies that α is integral over R_P. So we may assume that R is a local domain and $I \neq R$. Since A is a pseudo-simple extension, we have $\mathrm{grade} I > 1$, and hence there exists a regular sequence a, b in I. By assumption $a\alpha$ and $b\alpha$ are integral over R. Thus $a, b \in \sqrt{I_{[\alpha]}}$ by Lemma 11.4.11. But $\sqrt{I_{[\alpha]}}$ is a divisorial ideal, which implies that $\sqrt{I_{[\alpha]}} = R$ and hence $I_{[\alpha]} = R$. Thus $\varphi_\alpha(X) \in R[X]$. Therefore α is integral over R. ∎

Proposition 11.4.13 *Let $A = R[I\alpha]$ be a pseudo-simple extension of R defined by I and $\alpha \in K$. Assume that α is anti-integral over R. Then the following statements are equivalent:*

(a) *A is integral over R;*

(b) *α is integral over R;*

(*c*) $A = R$;

(*d*) $R = R[\alpha]$;

(*e*) $I \subseteq R :_R \alpha$.

Proof (*a*) $\Rightarrow$ (*b*) follows from Proposition 11.4.12. (*b*) $\Rightarrow$ (*d*) follows from Lemma 11.4.11. (*d*) $\Rightarrow$ (*c*), (*d*) $\Rightarrow$ (*b*), (*e*) $\Rightarrow$ (*c*), and (*b*) $\Rightarrow$ (*a*) are trivial. $\square$

Corollary 11.4.14 *Any proper pseudo-simple extension $R[I\alpha]$ of R defined by an ideal I and an anti-integral element α is not integral over R.*

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